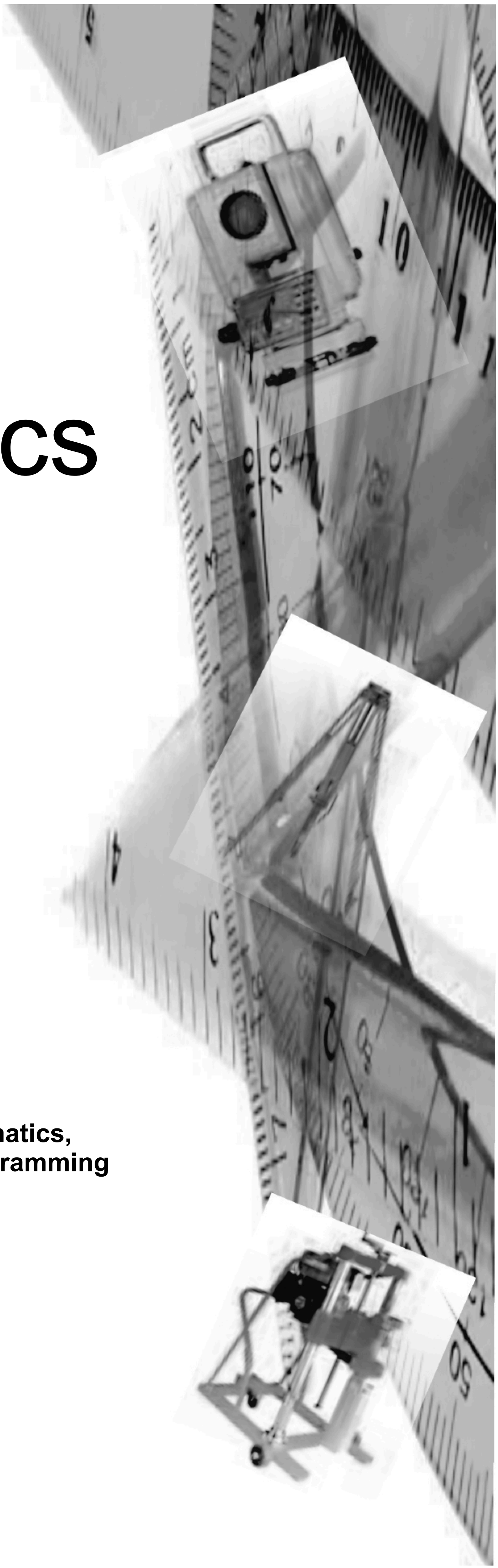
Mathematics



Unit 5

Sequences, Series, Financial Mathematics, Exponents, Logarithms and Linear Programming

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LESSON 1

Sequences

About this lesson

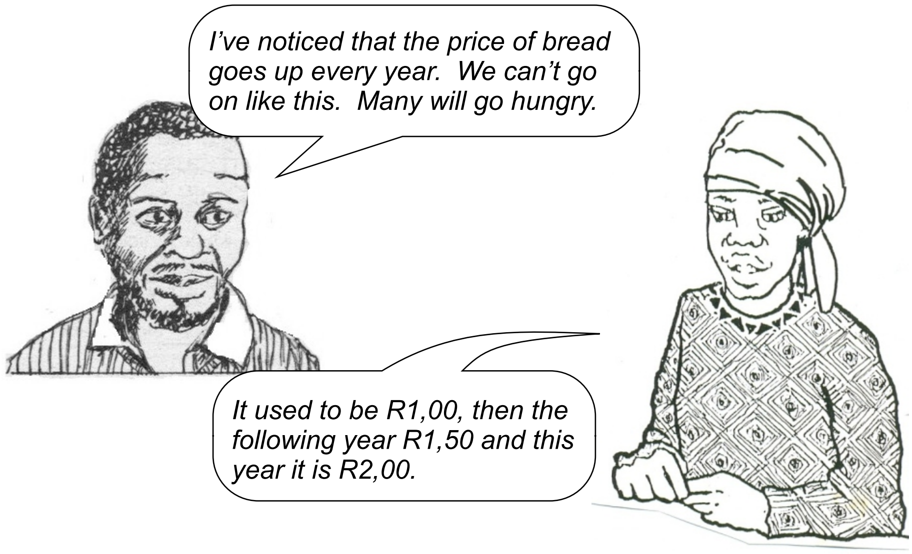
A very big task for mathematicians is the recognition of patterns. If you look up the word 'pattern' in the dictionary, you will see that the meanings of this word are 'order, arrangement, a system, consistency or a plan'. This is what mathematicians want to see. They want to see if a process has order and whether it follows a certain consistency or arrangement. When we say something follows a certain pattern we mean that it is predictable. If I say 2, 4, 6, 8, .., you can say what the following numbers are. That means you have recognised the pattern. You are able to say what the next number is.

In this lesson you will:

- work with number patterns
- identify sequences and their properties
- recognise arithmetic and geometric sequences
- order information

Number patterns that you know

Nobody likes to see prices of essential goods going up fast, while their salaries are going up more slowly, or not going up at all. This is a conversation I overheard while in a shop recently.



The discussion here is about something that two people have noticed as happening at a given time. They noticed a certain pattern in the changes in the price of bread.

One can easily change the statement of the first person to say: 'The price of bread usually goes up every year.' The statement of the second one can be changed to 'The price of bread was R1,00 then R1,50, then R2,00.'

These statements are both about price changes based on equal time periods. Both state that there is a starting point, the time of a fixed price of bread. This is followed by a certain period that passes without any price change, followed by the next change in price. This understanding can easily help make a general rule about price changes. This is what we call a pattern or sequence.

We say a pattern because we can easily predict when the next change in the price of bread will take place and the new price.

If we study the pattern, based on price, we can predict how much bread will cost at any given time.

Let's put the prices in a row or sequence, as follows: R1,00; R1,50; R2,00. If you follow the pattern, you will be able to predict the next price change to be R2,50 and then R3,00, and so on.

Try the following activity which involves similar patterns.

ACTIVITY 1

1. Try to find the pattern and state the next five terms.

   a) 1, 2, 3, 4, 5, ...
   b) 1, 3, 5, 7, 9, ...
   c) 2, 4, 6, 8, 10, ...
   d) 1, 5, 9, 13, 17, ...
   e) 5, 2, –1, –4, –7, ...
   f) 2, 4, 8, 16, 32, ...
   g) 1, 3, 9, 27, 81, ...

2. In each case, state what the rule is to form the next term in the pattern.

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Sequences and their properties

What is a sequence? A sequence is any **ordered** list of terms. The general form of a sequence is

T_1; T_2; T_3; T_4; T_5; ... T_k; ...

where T_1 = first term (read 'T one')
T_2 = second term (read 'T two')
T_3 = third term
T_4 = fourth term
T_5 = fifth term
...
...
T_k - general term (read 'T kay')
...

The number below the T (called a subscript) tells us the term in the sequence (i.e. 1^{st}, 2^{nd}, etc.). If we change the order of the terms then we get a different sequence.

A sequence may or may not have a last term. Sequences which do not show a pattern are called **random sequences**. In this lesson we shall deal with two types of sequences, **arithmetic** and **geometric sequences**.

*An **arithmetic sequence** is a sequence in which the difference between any two consecutive terms is a constant. This constant is called the common difference. The **common difference** is given by*

$$d = T_{k+1} - T_k$$

So $d = T_2 - T_1 = T_3 - T_2$, *etc*

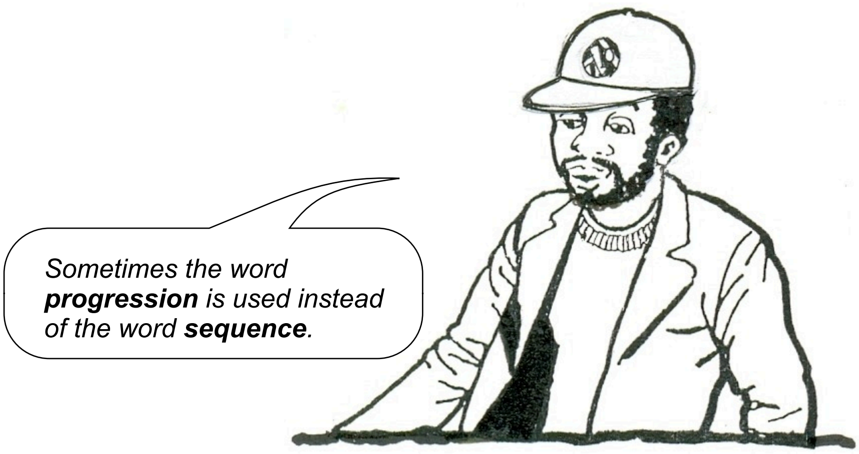
*A **geomatric sequence** is a sequence in which the ratio between any two consecutive terms is a constant. The common ratio is given by*

$$r = \frac{T_{k+1}}{T_k}$$

So

$$r = \frac{T_2}{T_1} = \frac{T_3}{T_2},$$

etc.



Arithmetic sequences

Look at the following sequence:

1; 4; 7; 10; 13; ...

This sequence has the first term $T_1 = 1$ and common difference $d = 3$. Can we write all the terms of this sequence using T_1 and d? Yes, we can. Note that

$T_1 = 1$
$T_2 = 4 = 1 + 3 = T_1 + d$
$T_3 = 7 = 1 + 6 = 1 + 2 \times 3 = T_1 + 2d$ and so on.

Try and work out T_4, T_5, and so on, using only T_1 and d. Note that the second term is $T_1 + 1 \times d$, the third term is $T_1 + 2 \times d$, the fourth term is $T_1 + 3 \times d$. What would be the eighth term? $T_1 + 7 \times d$ of course. Can you get the general rule?

Look at this: suppose the first term is the number a, then

$T_1 = a$
$T_2 = a + d$
$T_3 = a + 2d = a + (3 - 1)d$
$T_4 = a + 3d = a + (4 - 1)d$
....
$T_9 = a + 8d = a + (9 - 1)d$
....
$T_k = a + (k - 1)d$

T_k is called the general term of the sequence. This general term enables us to calculate any term of the sequence. For example, the 20th term of the sequence 1; 4; 7; 10; 13; ... is

$$\begin{aligned} T_{20} &= a + (20 - 1)d \\ &= 1 + 19x3 \\ &= 1 + 57 \\ &= 58 \end{aligned}$$

ACTIVITY 2

1. The general term of a sequence is given as $T_k = 5 + (k - 1)6$.

   a) What is the first term of the sequence?
   b) Find its second and third terms.
   c) What is its common difference?

2. The general term of a sequence is given as $T_k = 4 + 2k$.

   a) What is the first term of this sequence?
   b) What is its common difference?
   c) Which term of this sequence is equal to 46?

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ACTIVITY 3

1. Find the general term of each arithmetic sequence if

   a) the fourth term is –11 an the tenth term is –35.
   b) the third term is 16 and the sixth term is 37.
   c) the fourth term is $5x+8y$ and the tenth term is $11x+26y$.

2. The first three terms of an arithmetic sequence are given as $x+3; 3x+4; 4x+7$.

   a) Find the value of x.
   b) Find the general term of the sequence.
   c) Calculate the 50^{th} term.

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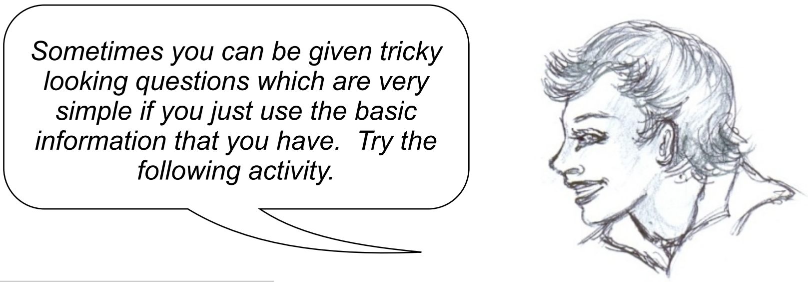

ACTIVITY 4

1. The sum of the first three consecutive terms of an arithmetic sequence is 12 and the product of the first two is 8. Find the sequence.

2. A consignment of 20 pieces of steel has been supplied to an engineering company. The quality inspector realises that the 11th item is 9 times the first item in mass, while the 7th item is 1 unit more than two times the mass of the third item. She is given information that the difference between any two consecutive masses is a constant. Find all these masses.

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Geometric sequences

You have already read that a geometric sequence is a sequence in which the ratio, r, between any two terms of the sequence, is a constant. We can find this common ratio by dividing a term by the one before it.

$$r = \frac{T_{k+1}}{T_k}$$

The first term is again $T_1 = a$

Then,

$$T_2 = a \cdot r = ar$$
$$T_3 = ar \cdot r = ar^2$$
$$T_4 = ar^2 \cdot r = ar^3$$
$$T_5 = ar^3 \cdot r = ar^4$$
$$\ldots\ldots\ldots\ldots$$
$$T_k = ar^{k-2} \cdot r = ar^{k-1}$$

If r = 1, all the terms will be equal.
If r = 0, the terms will be a; 0; 0; 0; ... and we can no longer use the definition:

$$r = \frac{T_{k+1}}{T_k}$$

In general we do not allow r = 0.

Before we continue, read this story.

One greedy businessman was invited to play a money game by a student who had managed to get a loan of R3 000 from a local bank. The student asked the man if he wanted to earn a lot of money quickly by playing a game. The man said, 'Yes'. The student went on to say: 'I will bring you R100 every day for 30 days. All you have to do on the first day is to give me R1,00, which you will double on the next day, and you keep on doubling the amount of the previous day, while I keep on giving you R100 every day.'

The man could not believe his luck. He agreed to make a signed statement of his agreement with the student. His wife warned him, but he would not listen. Typical.

On the first day the student brought R100, and the man gave the student R1,00. On the second day the student again brought R100, and the man gave him R2,00. Each day the student brought R100. By the 15th day the man was complaining. He was becoming bankrupt. On the 15th day the student brought him the 15th R100 note. The amount the student had brought him since they started the game was R1 500.

He had given the student the following sequence:

1; 2; 4; 8; ...

This can be written as $T_1 = a = 1, r = 2$.

$T_{15} = ar^{14} = 1 \cdot 2^{14} = 16\ 384$

The amount he had to give the student was R16 384. This amount was way beyond the amount the student had given him, and they were only half way. By the 21st day the student brought him his 21st R100 note. He had taken only R2 100 from the student. He had to give the student more than a million rand!

$$T_{21} = ar^{20} = 1 \cdot 2^{20} = 1\ 048\ 576$$

It was at this point that he had to give the student all his belongings. As a gesture of friendship, the student said he could keep his house. The problem the businessman had was that he did not know about geometric sequences.

First, the common ratio, r, does not have to be a whole number. For example,

$12; 6; 3; 1\frac{1}{2}; ...$ is a geometric sequence with $r = \frac{1}{2}$.

Secondly, the common ratio may be negative. Look at the following:

2; –4; 8; –16; 32; ... is a geometric sequence with $r = -2$.

It is easy to detect a geometric sequence with a negative ratio. The signs of the terms will change from one term to the next (this is called an alternating sequence).

Example

Determine the 5th and 10th terms of the geometric sequence:
2; 14; 98; ...

Solution

$T_1 = a = 2$ and $r = \frac{T_2}{T_1} = \frac{T_3}{T_2} = 7$

$T_k = ar^{k-1}$

$T_5 = ar^4 = 2 \times 7^4 = 4\ 802$

$T_{10} = ar^9 = 2 \times 7^9 = 80\ 707\ 214$

As you can see from the example, it is mainly the understanding of the formula for the general term that is necessary.

ACTIVITY 5

1. The 5th term of a geometric sequence is 1 875 and the common ratio is 5. Give the first three terms of this sequence.

2. Find the 6th term of a geometric sequence where the first term is 4 and the fourth is 0,864.

3. Determine the first term and the common ratio of a geometric sequence in which the fourth term is $24(x - y)^4$ and the seventh term is $192(x - y)^7$, $x \neq y$.

4. The first three terms of a geometric sequence are $x + 2$; $x - 2$ and x. Find the terms and their common ratio.

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Geometric or arithmetic?

How would you know whether what you are given is a geometric or arithmetic progression if you have not been told? This is quite easy. All you have to do is to check whether that given sequence satisfies the definitions of either the arithmetic or the geometric sequence. Look at the following example.

Example

State whether the following sequence is arithmetic or geometric:

$2; 4; 8; 16; 32; ...$

Solution

The first thing to check is whether this is an arithmetic sequence. An arithmetic sequence has a common difference between any two terms. Therefore, check the following:

$$T_2 - T_1 = 4 - 2 = 2$$
$$T_3 - T_2 = 8 - 4 = 4$$

This is enough to show that this is not an arithmetic sequence. Now check whether it is a geometric sequence. A geometric sequence has a common ratio between any two elements. Proceed as follows:

$$\frac{T_2}{T_1} = \frac{4}{2} = 2$$
$$\frac{T_3}{T_2} = \frac{8}{4} = 2$$

This is enough to show that the sequence is a geometric sequence.

ACTIVITY 6

State whether each of the following sequences is arithmetic or geometric.

a) $3;\ -6;\ 12;\ -24; ...$
b) $-15;\ -9;\ -3;\ 3; ...$
c) $x;\ bx^2;\ b^2x^3;\ b^3x^4; ...$

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Other sequences

A sequence does not have to be arithmetic or geometric. Any ordered list is called a sequence. Some sequences form obvious patterns, others do not. Some sequences have a formula for the general term, many do not.

ACTIVITY 7

For each of the following sequences, try to find a pattern and then write down the next three terms of the sequence. If possible, write down a formula for the general term:

a) $1;\ 4;\ 9;\ 16; ...$
b) $1;\ \frac{1}{2};\ \frac{1}{3};\ \frac{1}{4}; ...$
c) $1;\ 11; 111;\ 1111; ...$
d) $3;\ 3{,}1;\ 3{,}14;\ 3{,}141;\ 3{,}1415; ...$

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ACTIVITY 8

Write down the first three terms and the 20th term of each of the following sequences:

a) $T_k = \dfrac{k}{k+1}$

b) $T_k = (-1)^k \text{ x } k$

c) $T_k = \dfrac{(-1)^{k+1}}{k^2}$

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Summary

In the definitions you learnt about a common difference in the case of an arithmetic sequence, and a common ratio in the case of a geometric sequence. These are working definitions and are important to remember. If you do not know them you are not going to be able to solve problems on sequences.

You have also been introduced to problems on sequences. Remember, sequences are sometimes called progressions. You should be able to list terms of an arithmetic and geometric sequence. The **first term** and the **common difference** are crucial for arithmetic sequences. The **first term** and the **common ratio** are crucial for geometric sequences.

Finally, remember that not all sequences are either arithmetic or geometric. There are many different kinds.

If you are satisfied that you have understood this lesson, check yourself in the following exercises.

arithmetic sequence
$d = T_{k+1} - T_k$ *where*
d = common difference
and $T_k = a + (k-1)d$
is the general term,
where $T_1 = a$.

geometric sequence
$r = \dfrac{T_{k+1}}{T_k}$ *where*
r = common ratio
and $T_k = ar^{k-1}$
is the general term,
where $T_1 = a$.

CHECKLIST

Are you able to:

- recognise various number patterns
- work with arithmetic and geometric sequences
- order information
- list terms of a sequence

SELF-CHECK EXERCISE

1. Find the first three terms of the following arithmetic sequences:

   a) the fifth term is -1 and the eleventh term is -19.

   b) the fourth term is -4 and the seventh term is $-6m-16$.

2. Find the first three terms of the following geometric sequences:

   a) The third term is 3, the sixth term is $-\frac{1}{9}$.

   b) The fifth term is $\frac{27}{7}$, the eight term is $\frac{729}{7}$.

   c) The ninth term is $(x^9 + 2x^8b)$, the 13th term is $(x^{13} + 2x^{12}b)$.

3. Determine the fifth and the fiftieth term of the following arithmetic sequences:

   a) $1;\ 5;\ 9;\ldots$
   b) $a=-22,\ d=6$
   c) $\mathrm{a}=100,\ T_4=85$
   d) $x;\ x-y;\ x-2y;\ldots$

4. Determine the fifth and fiftieth term of the following geometric sequences:

   a) $1; 3; 9;\ldots$

   b) $x+y;\ x^2-y^2;\ x^3-x^2y-xy^2+y^3;\ldots$

   c) $1;\ \frac{1}{2};\ \frac{1}{4};\ldots$

   d) $T_8 = 3 \cdot 10^9$
      $T_{12} = 3 \cdot 10^{13}$

5. $2; x; y$ are the first three terms of an arithmetic sequence, and $2; x-1; y$ are the first three terms of a geometric sequence. Calculate x and y.

6. a) 963 is the nth term of the arithmetic sequence 3; 11; 19;... Find n.
   b) Is 546 a term of the geometric sequence 7; 14; 28;...?
   c) What is the 50th odd number?

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LESSON 2

Series

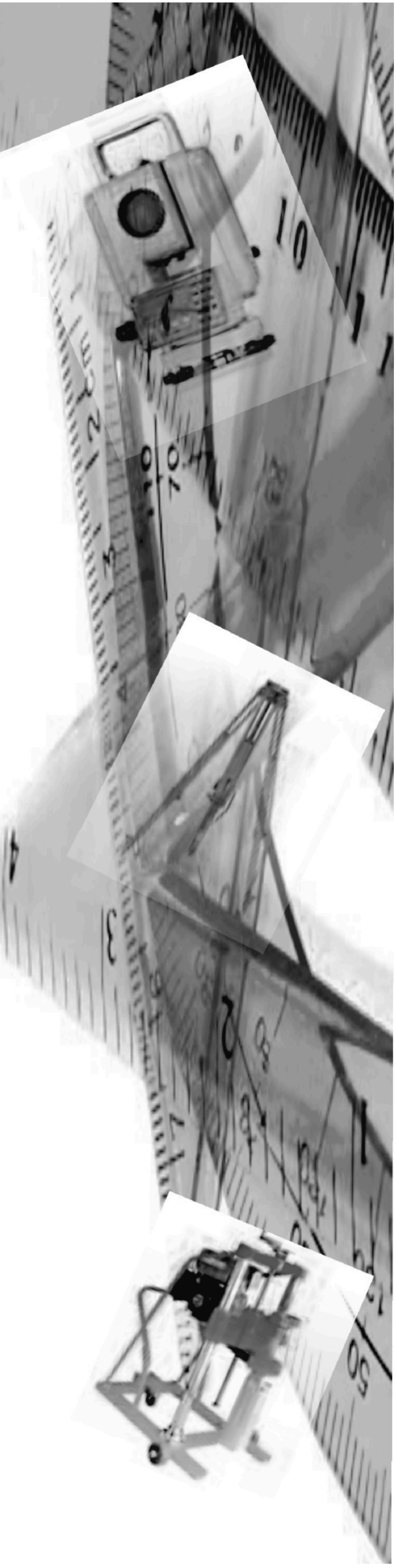
About this lesson

Suppose you were very good at working out sequences and a friend asked you to work out the sum of the natural numbers from one to a thousand, how would you do that?

This lesson is about sums of sequences. We shall also look at compound interest. You need to know the definitions and terms that you learnt in the previous lesson really well.

In this lesson you will:

- revise sequences
- learn about formulae to find arithmetic and geometric series
- learn about sigma notation



Revision of sequences

- A **sequence** is any ordered list $T_1;\ T_2;\ T_3;...;\ T_k;...$
- The elements of the list are called terms.
  T_1 is called the first term of the sequence (often denoted *a*).
  T_2 is called the second term of the sequence.
  T_k is called the k[th] or general term of the sequence.
- If a sequence has a last term (often denoted *l*) then we say the sequence is finite. If there is no last term then the sequence is said to be infinite.
- If the order of the terms is changed then we obtain a new sequence.
- If $T_{k+1} - T_k = d$, a constant called the common difference, then the sequence is arithmetic and $T_k = a + (k-1)d$ where $T_1 = a$.
- If $\dfrac{T_{k+1}}{T_k} = r$, a constant called the common ratio, then the sequence is geometric and $T_k = ar^{k-1}$ where $T_1 = a$.
- Not all sequences are either arithmetic or geometric, but these are the sequences we are most interested in for this course.
- The choice of the letter *k* for the general term is arbitrary. We could just as well write T_n or T_m or...
  So then we get $T_n = a + (n-1)d$ and $T_n = ar^{n-1}$, etc.

Now that you've revised sequences, you can go on to a new topic, series.

Arithmetic Series

We need to add all the natural numbers from 1 to 1000 (an arithmetic sequence with $a = 1$ and $d = 1$),
i.e. 1 + 2 + 3 + 4 + ... + 998 + 999 + 1000.

This could be done on a calculator (or even mentally) but would be very tedious. Let's start by trying to find a method for the sum of the natural numbers from 1 to 10 and then see if we can generalise the method. (This is a very typical approach used by professional mathematicians: first try to solve a simpler version of the problem and then generalise the solution.)

Let $S_{10} = 1+2+3+4+5+6+7+8+9+10$ ①

(S stands for 'sum' and S_{10} stands for the sum of the **first** 10 terms of our sequence.)

Because there is a finite number of terms, we can write the sum in reverse order and still obtain the same answer:

$$S_{10} = 10+9+8+7+6+5+4+3+2+1 \quad ②$$

Adding ① and ② gives:

$$2S_{10} = 11+11+11+11+11+11+11+11+11+11 \text{ (10 terms)}$$

$$= 10.11 \text{ (number of terms x [first term + last term])}$$

$$\therefore S_{10} = \frac{10.11}{2} = 55$$

It seems pretty obvious that this method will also work for our original question:

$$S_{1000} = 1+2+3+\ldots+998+999+1\,000 \quad ①$$

and $S_{1000} = 1\,000+999+998+\ldots+3+2+1$ ②

$$① + ②: \; 2S_{1000} = 1\,001+1\,001+\ldots+1\,001+1\,001 \text{ (1 000 terms)}$$

$$= 1\,000.1\,001$$

$$\therefore S_{1000} = \frac{1\,000.1\,001}{2} = 500\,500$$

Notice that in each case the answer is given by

$$S_n = \frac{n.[T_1 + T_n]}{2}$$

where: n is the number of terms

T_1 is the **first** term
T_n is the nth or **last** term.

If $T_1 = a$ and $T_n = l$, we have $S_n = \frac{n}{2}\{a+l\}$

Let's prove this:
Suppose $T_1; T_2; \ldots; T_n$ is an arithmetic sequence with

$T_1 = a,\ T_n = l$ and common difference d, then the sum of the first

n terms $T_1 + T_2 + \ldots + T_n$ is given by $S_n = \frac{n}{2}\{a+l\}$

Proof:

$$S_n = T_1 + T + ... + T_n$$
$$= a + (a + d) + ... + (a + (n - 1)d) \quad ①$$
$$\text{and } S_n = T_n + T_{n-1} + ... + T_1$$
$$= l + (l - d) + ... + (l - (n - 1)d) \quad ②$$
$$①+②:\ 2S_n = \{a + l\} + \{a + l\} + ... + \{a + l\} \quad (n \text{ terms})$$
$$= n.\{a + l\}$$
$$\therefore \boxed{S_n = \frac{n}{2}\{a + l\}}$$

1. When we reverse the order to obtain ② we have to subtract d from l:

   $T_n = l,\ T_{n-1} = l - d,\ T_{n-2} = l - 2d, ..., T_n = l - (n - 1)d$

   (e.g. for the sequence 1, 2, 3, 4, 5 we have:

   $T_1 = 1,\ T_2 = 1 + 1,\ T_3 = 1 + 2,\ T_4 = 1 + 3,\ T_5 = 1 + 4$

   but starting from the back:

   $T_5 = 5,\ T_4 = 5 - 1,\ T_3 = 5 - 2,\ T_2 = 5 - 3,\ T_1 = 5 - 4$)

2. Since $l = T_n = a + (n - 1)d$, we can write

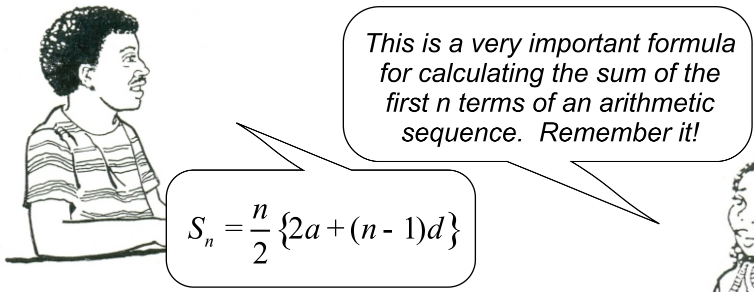
   $$S_n = \frac{n}{2}\{a + l\}$$
   $$= \frac{n}{2}\{a + a + (n - 1)d\} \quad \text{(substitute for } l\text{)}$$
   $$\therefore \boxed{S_n = \frac{n}{2}\{2a + (n - 1)d\}}$$

3. You may be asked to prove, or derive these formulas in an examination.

4. If the **sequence** is $T_1;\ T_2;\ ...;\ T_k$ then the **series** is $T_1 + T_2 + ... + T_k$.

   A sequence is a list and a series is a sum, e.g. if the sequence is 1; 4; 7; 10; 13 then the associated series is 1 + 4 + 7 + 10 + 13.

5. The formula $S_n = \frac{n}{2}\{a + l\}$ is only useful when we know the last term. Generally $S_n = \frac{n}{2}\{2a + (n - 1)d\}$ is more useful.



Now do the following activity.

ACTIVITY 1

1. Find the sum of the first 100 natural numbers.

2. Find the sum of the first hundred even natural numbers.

3. Find the sum of the first hundred odd numbers.

4. Find the sum of the first twenty terms of the following sequence:

   $1;\ 2\frac{1}{3};\ 3\frac{2}{3};\ 5;...$ (i.e. of the series $1 + 2\frac{1}{3} + 3\frac{2}{3} + 5 + ...$)

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Geometric Series

We can also get the sum of the first n terms of geometric sequence. What formula can we use now? Look at the following example.

Example
Find the sum of the first ten terms of the geometric sequence 3; 9; 27; ... (i.e. the series $3 + 9 + 27 + ...$ to 10 terms).

Solution
Suppose we try to use the formula we used for arithmetic series: $a = 3$, $r = 3, n = 10$.

$$T_n = ar^{n-1}$$
$$\therefore T_{10} = l = 3.3^9 = 3^{10}$$

$$S_n = \frac{n}{2}(a + l)$$
$$\therefore S_{10} = \frac{10}{2}(3 + 3^{10})$$
$$= 5 \times (3 + 59\ 049)$$
$$= 295\ 260$$

Is this true? Let us check this by adding the numbers step by step using a calculator. The series is

3 + 9 + 27 + 81 + 243 + 729 + 2 187 + 6 561 + 19 683 + 59 049 = 88 572

This shows that the formula for arithmetic series does not work here. Instead, we do the following:

$$S_{10} = 3 + 9 + 27 + ... + 19\ 683 + 59\ 049 \qquad ⑤$$

Now multiply all the terms in the series by 3, the common ratio:

$$3S_{10} = 9 + 27 + 81 + ... + 59\ 049 + 177\ 147 \qquad ⑥$$

Subtracting ⑤ from ⑥ : $3S_{10} - S_{10} = 177\ 147 - 3$

Can you see that all the terms from 9 to 59049 cancel each other?

So $\quad 2S_{10} = 177\ 144$

$$\therefore S_{10} = \frac{177\ 144}{2}$$

$$= 88\ 572$$

Hey! It's worked! Let's now check whether this works with other examples.

Example

Find the sum of the following geometric series:

$1 + 4 + 16 + 64 + 256$.

Solution

Here $a = 1$, $r = 4$ and $n = 5$.

$$S_5 = 1 + 4 + 16 + 64 + 256 \qquad ①$$

$$4S_5 = 4 + 16 + 64 + 256 + 1\ 024 \qquad ② \qquad \text{(multiply by } r\text{)}$$

$$4S_5 - S_5 = 1\ 024 - 1 \qquad ①–②$$

$$3S_5 = 1\ 023$$

$$S_5 = 341$$

Did you notice that in each case we ended up with:

$$S_n = \frac{ar^n - a}{r - 1}$$

Let's prove this:

Suppose $T_1;\ T_2;...;\ T_n$ is a geometric sequence with $T_1 = a$ and common ratio $r \neq 1$, then the sum S_n of the first n terms $T_1 + T_2 + ... + T_n$ is given by:

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

Proof:

$$S_n = a + ar + ar^2 + ar^3 + ... + ar^{n-2} + ar^{n-1} \qquad ①$$

Now, multiply S_n by r, to get:

$$rS_n = ar + ar^2 + ar^3 + ar^4 ... + ar^{n-1} + ar^n \qquad ②$$

② - ① $rS_n - S_n = ar^n - a$

$S_n(r-1) = a(r^n - 1)$

$$\therefore S_n = \frac{a(r^n-1)}{r-1}$$

Note: ① - ② gives: $S_n - rS_n = a - ar^n$

$S_n(1-r) = a(1-r^n)$

$$\therefore S_n = \frac{a(1-r^n)}{1-r}$$

This formula is easier to use when $r < 1$, but either will give the correct answer.

Example

Find $3 + \frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \frac{3}{16}$

This is a geometric series with $a = 3,\ r = \frac{1}{2},\ n = 5$

$$S_n = \frac{a(1-r^n)}{1-r} \quad \text{or} \quad S_n = \frac{a(r^n-1)}{r-1}$$

$$\therefore S_5 = \frac{3\left(1-\frac{1}{32}\right)}{1-\frac{1}{2}} \qquad S_5 = \frac{3\left(\frac{1}{32}-1\right)}{\frac{1}{2}-1}$$

$$= 3 \cdot \frac{31}{32} \cdot \frac{2}{1} \qquad = 3.\left(\frac{-31}{32}\right).\left(\frac{-2}{1}\right)$$

$$= \frac{93}{16} = 5\frac{13}{16} \qquad = \frac{93}{16} = 5\frac{13}{16}$$

ACTIVITY 2

1. Find how much you will need to pay in total if you were asked to pay instalments in this manner: R2,00 in the first month, which is then doubled every month, for twenty four months. How much is the last instalment?

2. Find the sum of the first 15 terms of the geometric progression:

   $6;\ 3;\ \frac{3}{2};\ ....$

3. The second term of a geometric series is 6 and the fifth term is 162. Find the sum of the first 20 terms.

ANSWERS ON PAGE 77

Infinite geometric series

So far all the sequence and associated series we have looked at have had a finite number of terms. Generally speaking, all sequences are actually infinite, i.e. have infinitely many terms.

Examples:

a) 1; 2; 3; 4; 5; ... (the natural numbers) is an infinite arithmetic sequence ($a = d = 1$).

b) $1;\ \frac{1}{2};\ \frac{1}{4};\ \frac{1}{8};...$ is an infinite geometric sequence ($a = 1,\ r = \frac{1}{2}$).

Unless $d = 0$, the sum of an infinite arithmetic sequence is always $-\infty$ or ∞ (these symbols are not numbers; they simply mean the sum gets bigger and bigger in value).

If $r \geq 1$, the sum of a geometric series is also $-\infty$ or ∞, depending on a.
If $r \leq -1$, a geometric series oscillates and can do very strange things (experiment with $a = 1, r = -1$ and also with $a = 1, r = -2$).

However, if $-1 < r < 1$, then a geometric series gets closer and closer and closer to a constant value as we add more and more terms.

Example:

Consider the infinite geometric series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + ...$ with $a = 1$ and $r = \frac{1}{2}$.

$$S_1 = 1$$

$$S_2 = 1 + \frac{1}{2} = 1\frac{1}{2} = 1,5$$

$$S_3 = 1 + \frac{1}{2} + \frac{1}{4} = 1\frac{3}{4} = 1,75$$

$$S_4 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = 1\frac{7}{8} = 1,875$$

$$S_5 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = 1\frac{15}{16} = 1,9375$$

$$S_6 = 1\frac{31}{32} = 1,96875$$

$$S_7 = 1\frac{63}{64} = 1,984375\ ...$$

$$S_{10} = 1\frac{511}{512} = 1,998046875$$

It would appear that the more terms we add, the closer the answer gets to 2. We say that the series **converges** to 2. Can you see that the answer can never be greater than or even actually equal to 2? Look at the proper fraction part in each case. What do you notice?

It can be shown that:
If $T_1;\ T_2;...$ is a geometric sequence with $T_1 = a$ and $-1 < r < 1$ (this is really important), then the 'sum to infinity' is given by

$$S_\infty = \frac{a}{1-r}$$

Example

Find the sum to infinity of the geometric series $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + ...$

Solution

$$a = 1,\ r = \frac{1}{2} \quad \therefore\ S_\infty = \frac{a}{1-r} = \frac{1}{1-\frac{1}{2}} = 2$$

ACTIVITY 3

1. Find the sum to infinity of the geometric series:
   $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + ...$
2. Convert the recurring decimal 0,3333... to a common fraction.
3. For which values of x will this geometric series converge:
   $x + 2x^2 + 4x^3 + ...$ What is its sum?

ANSWERS ON PAGE 78

Sigma notation

Many books of mathematics and statistics use a short form when writing series. This short form is widely used in mathematics. The symbol is for a sum of terms of any series. The notation is called **sigma**, written asΣ.

Σ stands for summation, i.e. 'please add'.

For example, if you are given the following series:

$S_{10} = 1 + 2 + 3 + 4 + ... + 10$. This is a series of natural numbers from 1 to 10. You can write this as follows: $\sum_{n=1}^{10} n$

This simply reads 'the sum of all numbers n, where n is from 1 to 10'.

We can write many types of series in this form. These can be either arithmetic or geometric series. Look at the following examples:

Example

$2+5+8+...+32$

Solution

This is an arithmetic series that has the first term $a = 2$ and the common difference $d = 3$. The last term is 32. The first task is to find how many terms the series has.

The last term of the series is 32. We will use the formula

$$T_n = a+(n-1)d = 2+(n-1)3 = 3n-1$$

$$32 = 2+3(n-1)$$

$$3n-1 = 32$$

$$n = 11$$

We can now write the sum of the series as follows:

$$\sum_{n=1}^{11}(3n-1) \quad \text{or even} \quad \sum_{k=1}^{11}(3k-1) \quad \text{(the letter does not matter)}$$

Example

Find the sum of the first ten terms of the sequence 3; 6; 12; 24; ...

Solution

This is a geometric series where a = 3, r = 2, n = 10. Everything is given here. The sum of the series can be written as follows:

$$\sum_{n=1}^{10} ar^{n-1} = \sum_{n=1}^{10} 3.2^{n-1} = \frac{3(2^{10}-1)}{2-1} = 3\ 069$$

Now practise by doing the following activities.

All the formulas derived previously can be written in Σ notation:

- *For an arithmetic sequence with first term a and common difference d:*

$$S_n = \sum_{i=1}^{n}\{a+(i-1)d\} = \frac{n}{2}\{2a+(n-1)d\}$$

- *For a geometric sequence with first term a and common ratio r(r≠1):*

$$S_n = \sum_{i=1}^{n} ar^{i-1} = \frac{a(r^n-1)}{r-1}$$

- *For a geometric sequence with first Term a and common ratio r, where −1 < r < 1,*

$$S_\infty = \sum_{i=1}^{\infty} ar^{i-1} = \frac{a}{1-r}$$

ACTIVITY 4

1. Expand (i.e. write out without the Σ sign) each of the following series:

a) $\sum_{n=1}^{5} n^2$

b) $\sum_{k=1}^{10} (2k-1)$

c) $\sum_{r=0}^{5} \frac{1}{r+1}$

d) $\sum_{n=1}^{12} 2$

e) $\sum_{n=1}^{\infty} \frac{1}{2^n}$

2. Write down in $\sum$ notation and calculate the sum in each case:

a) $1+5+9+...$ (twenty terms)
b) $4+7+10+...+31$
c) $10+5+2,5+...$ (ten terms)
d) $2+5+12,5+...$ (ten terms)
e) $1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+...$

ANSWERS ON PAGE 78

You may have realised that there is nothing new in this. The summation sign does not change the method for the calculations of sums of series. You will use the $\sum$ sign again in the next unit.

Summary

At the beginning of this lesson we used a few questions to revise sequences. This helped you to revise the previous lesson.

You learnt the formula that is used to find:

- sums of arithmetic series $S_n=\frac{n}{2}[2a+(n-1)d]=\frac{n}{2}\{a+l\}$
- geometric series $S_n=\frac{a(r^n-1)}{r-1}$ or $S_n=\frac{a(1-r^n)}{1-r}$
- infinite geometric series $S_\infty=\frac{a}{1-r}$ if $-1<r<1$

You also found out about the summation sign, **sigma**, which we use in conjunction with the formula for the general term. There is not much that changes when we use the summation sign. It only helps to write series in a short form.

CHECKLIST

Are you able to:
- work with sequences
- use formulae to find arithmetic and geometric series
- use the sigma notation to write series in a short form

SELF-CHECK EXERCISE

1. Look at this formula: $S = \frac{a(r^n - 1)}{r - 1}$. Why do we say r cannot be equal to 1?

2. Express in $\sum$ notation and find the sums of the following series:
   a) $3 + 8 + 13 + ... + 63$
   b) $3 + (-6) + 12 + (-24) + ...$ (ten terms)
   c) $9 + 6 + 4 + ...$(infinitely many terms)

3. Show that

   a) $\sum_{i=1}^{n} 1 = n$
   b) $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
   c) $\sum_{i=1}^{n} (2i - 1) = n^2$

4. Write the recurring decimal 0,343434... in the form $\frac{a}{b}$ where a and b are positive integers.

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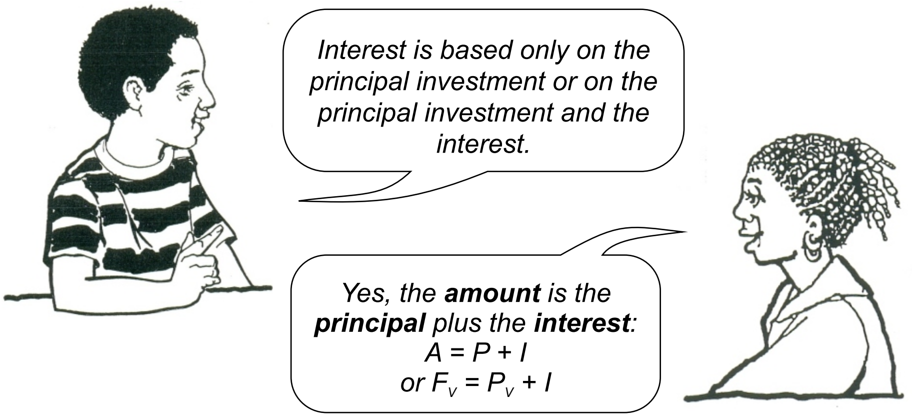
LESSON 3

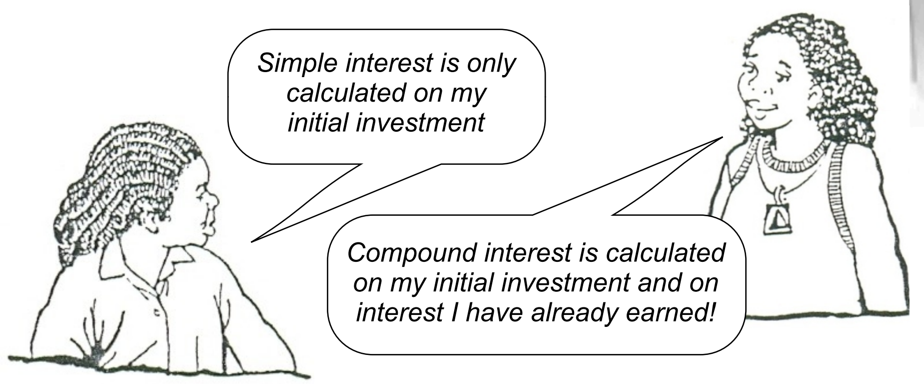
Financial Mathematics

About this lesson

When you borrow money from a bank or money lender you pay an amount to the bank for the use of the money. Also, when you invest money in a bank, i.e. you lend your money to the bank, then the bank pays you an amount for the use of your money. In each case, the amount paid for the use of money is called interest.

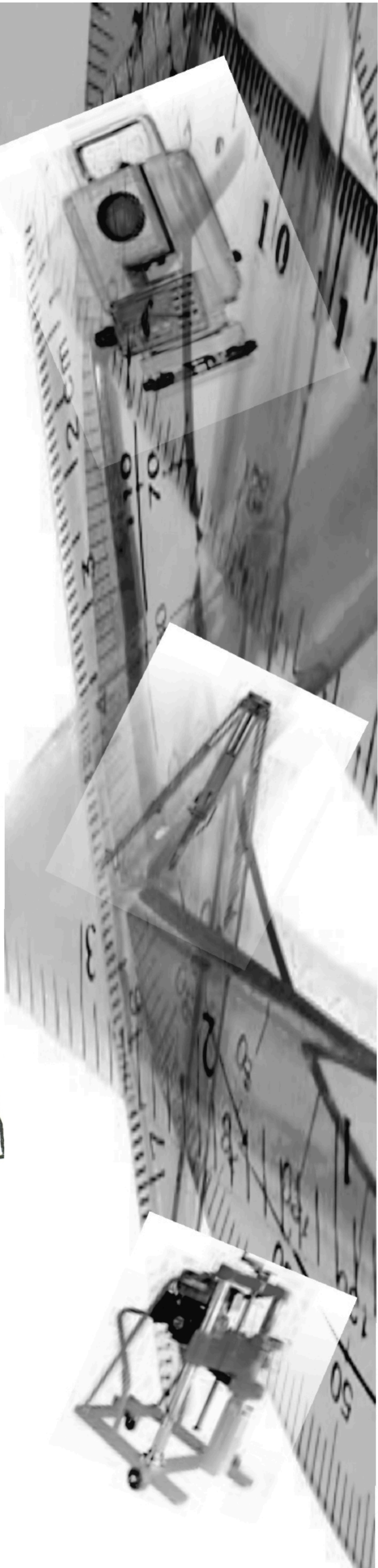
One usually invests a specific amount of money for a specific period of time.





In this lesson you will:

- learn about simple and compound interest and how to calculate interest
- use formulae to calculate interest
- learn about depreciation
- work with nominal and effective rates
- understand annuities and repayments
- understand how exchange rates work



Interest Calculations

Example:
I have R1 000 to invest. How much will I get after 3 years if:

a) I get 4% simple interest per year?
b) I get 4% compound interest per year?

Solution:

The principal $P = \text{R}1\ 000$
The interest rate $i = 4\% = 0{,}04$
The period $n = 3$ years

a) Simple interest:

For the 1st year the interest $= 4\%$ of R1 000
$= 0{,}04 \times \text{R}1\ 000 = \text{R}40$
$\therefore$ the amount after 1 year is $A = \text{R}1\ 000 + \text{R}40 = \text{R}1\ 040$

For the 2nd year the interest $= 4\%$ of R1 000
$=$ R40 (same)
$\therefore$ the amount after 2 years is $A = \text{R}1\ 000 + 2 \times \text{R}40 = \text{R}1\ 080$

For the 3rd year the interest is $= 4\%$ of R1 000
$= 0{,}04 \times \text{R}1\ 000 = \text{R}40$ (same again)
$\therefore$ the amount after 3 years is $A = \text{R}1\ 000 + 3 \times \text{R}40 = \text{R}1\ 120$

interest:
*amount paid for the use of money. One denotes interest as **I** and is usually calculated as a percentage of the loan or investment called the **interest rate**, denoted **i***

principal:
*amount of money initially borrowed or invested. Also called the **present value** and is denoted as **P** or* P_v

period:
*the amount of time the principal is invested and is denoted as **n***

amount or future value:
*the final value of the money after the interest has been added to the principal, denoted **A** or* F_v

simple interest:
based only on the initial investment also called the principal

compound interest:
interest based on the principal and interest already earned

Here, all the interest is added at the **end** of the 3 years. Notice that the final amount is:

$$A = \text{R1 000} + 3 \times 0{,}04 \times \text{R1 000}$$
$$\therefore A = \text{R1 000}\ (1 + 3 \times 0{,}04)$$

In general: $A = P(1 + n.i)$ - S.I. formula

b) Compound interest:

For the 1st year the interest = 4% of R1 000
= 0,04 × R1 000
= R40

For the 2nd year, the 1st year's interest is added **before** the interest is calculated, so now the interest is calculated on R1_000 + R40 = R1_040
and the interest for the 2nd year = 4% of R1_040
= 0,04 × R1 040
= R41,60 (R1,60 extra!)

∴ the amount after 2 years is = R1 040 + R41,60
= R1 081,60

For the 3rd year, the interest is calculated on the principal (R1 000) plus all the interest already earned = 4% of R1 081,60
= 0,04 × R1 081,60
= R43,26 (to 2 dec.)

∴ the amount after 3 years is = R1 081,60 + R43,26
= R1 124,86 (R4,86 extra)

Here the interest is added each year and we get interest on the principal and also on the interest already earned. Notice that the final amount after each year is:

$$A_1 = P + Pi = P(1+i)^1$$
$$A_2 = P(1+i) + P(1+i)i = P(1+i)(1+i) = P(1+i)^2$$
$$A_3 = P(1+i)^2 + P(1+i)^2 i = P(1+i)^2(1+i) = P(1+i)^3$$

In general: $A = P(1+i)^n$ - C.I. formula

To summarise:

P = principal (initial deposit or loan)
i = % interest rate per year (or per annum, written p.a.)
n = period (number of years)
A = amount (principal and interest)

S.I. formula: $A = P(1 + n.i)$ (interest on P only)

C.I. formula: $A = P(1+i)^n$ (interest on P and previous interest)

ACTIVITY 1

Suppose you deposited R500 in a bank at 12% interest per annum. How much will you expect to get after 5 years if the interest is:

a) simple interest?

b) compound interest?

ANSWERS ON PAGE 80

Other ways in which these formulas can be used

It is important to mention here that our formulas do not always have to deal with banks and loans, they can be used in other ways. Let's take a look at such an example.

ACTIVITY 2

In 1968 the population in South Africa was 25 million. The population growth averages 1,8% per year. Calculate the approximate population in the year 2000 if we assume that the growth remains at a constant rate of 1,8% per year. Approximate the answer to the nearest million.

ANSWERS ON PAGE 81

Compounded interest more than once per annum

In many cases interest is compounded a number of times per year. Suppose interest is compounded every 6 months, i.e. twice per year, the rate for 6 months will then become $\frac{i}{2}$.

If this interest is compounded twice a year over five years, then the number of times this will be done will be 5×2 times. If interest is compounded t times per year, the rate per period will then become $\frac{i}{t}$.

The number of times this is done in n years will the be nt times. The compound interest formula then becomes:

$$A = P(1 + \frac{i}{t})^{nt}$$

where A = total amount

P = principal deposited initially

i = % interest rate per annum

n = number of years

and t = number of times a year interest is compounded

You may find this information useful when you borrow money from a bank or invest money in a bank.

ACTIVITY 3

I deposit R1 000 into an account at **Savings Bank**. Calculate how much I receive from the bank after 10 years if the bank pays 5% p.a.

a) compounded annually
b) compounded 6 monthly
c) compounded monthly
d) compounded daily

ANSWERS ON PAGE 81

Simple and Compound Decay

Instead of increasing (growing or earning interest) there are situations where amounts or quantities decrease (decay or depreciate). Typical examples are: water in a lake will usually decrease by a certain percentage every day because of evaporation (this would be compound decay); the value of an asset, like a motor car, will decrease by a fixed amount every year. This would be simple depreciation.

depreciate:
decrease in value

Simple or Straight Line Depreciation

As for simple interest, this is based only on the initial value and is the same for every time period.

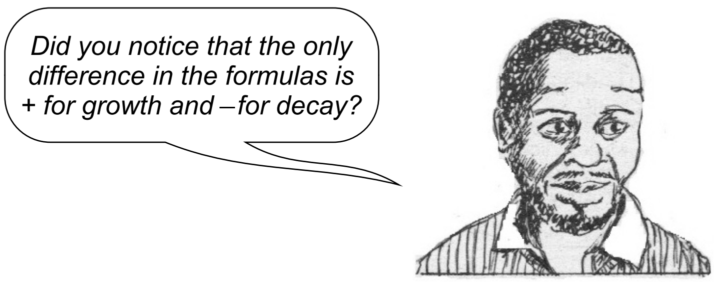
$$A = P(1 - in)$$

Compound or Reducing Balance Depreciation

As for compound interest, the depreciation is first subtracted before the next calculation is performed on the amended (reduced) balance.

$$A = P(1 - i)^n$$

When the period is not one year the formula is again: $A = P(1 - \frac{i}{t})^{nt}$



ACTIVITY 4

A woman buys a car for R45 000. What is the value of the car at the end of four years, if it depreciates:

a) at 4% p.a. on a straight line basis?
b) at 4% p.a. calculated half yearly on a reducing balance basis?

Give your answers correct to the nearest rand.

ANSWERS ON PAGE 81

Nominal and Effective Rates

When a financial institution offers you an interest rate of, say, 4% p.a. compounded monthly, this is an example of a **nominal** interest rate. However, because you earn interest on interest every month after a year you will have earned more than 4%. The actual interest rate which will give the same amount on an annual basis (instead of monthly) is called the **effective** interest rate.

Example

If you invest R1000 for 12 months (1 year) at 4% p.a. compounded monthly, by the end of the year you will have:

$$A = 1\ 000(1+\frac{0,04}{12})^{12}$$

$$= R1\ 040,74$$

*4% p.a. compounded monthly is the **nominal** rate;*

*4,074%p.a. compounded annually is the **effective** rate; it is the rate by which your investment actually increases over the period.*

To earn the same amount if the interest is only added at the end of the period would require a higher interest rate j, say:

$$1\ 040,74 = 1\ 000(1+j)$$

$$\therefore 1+j = 104\ 074 \text{ x } 1+j = 1,040\ 74$$

$$j = 0,040\ 74$$

$$= 4,074\%$$

ACTIVITY 5

1. R5 000 is deposited into a bank account offering 5,25% p.a. compounded daily for 1 year.

   a) What is the nominal interest rate?
   b) Calculate the amount in the bank at the end of the year.
   c) What is the effective annual interest rate?

2. Convert each of the following nominal rates to effective interest rates per annum.

   a) 9% p.a. compounded every 6 months
   b) 9% p.a. compounded every 3 months
   c) 9% p.a. compounded monthly

ANSWERS ON PAGE 82

Annuities, Repayments and Sinking Funds

In real life situations we usually have to pay equal regular amounts to an institution (bond repayments, pension or provident funds, etc) rather than just one initial deposit or payment. The calculations and formulas used for this are a very good application of geometric series, so perhaps you should revise that section before continuing with this one.

Future Value of an Annuity, F_V:

Example:

If you put R300 at the beginning of each month into an account paying 5% p.a. compounded monthly, how much will you have:

a) immediately after the 36th deposit?

b) at the end of 3 years?

Solution:

These questions are best solved with the aid of a 'time line':

T_0	T_1	T_2	T_3 …		T_{35}	T_{36}
R300	R300	R300	R300	i = 5% = 0,05	R300	R0

T_0 represents the start of the annuity
T_1 represents the end of the first period
T_2 represents the end of the second period, etc.

a) Immediately after the 1st payment $F_V = \text{R}300$

Immediately after the 2nd payment $F_V = R300 + R300(1+\frac{0,05}{12})^1$

1st payment + interest for 1 month

Immediately after the 3rd payment

$$F_V = R300 + R300(1+\frac{0,05}{12})^1 + R300(1+\frac{0,05}{12})^2$$

2nd payment + interest for 1 month; 1st payment + interest for 2 months

Immediately after the 36th payment

$$F_V = R300 + R300(1+\frac{0,05}{12})^1 + R300(1+\frac{0,05}{12})^2 + \ldots + R300(1+\frac{0,05}{12})^{35}$$

This is a geometric series with $a = 300,\ r = (1+\frac{0,05}{12}),\ n = 36$

$$F_V = \frac{a(r^n - 1)}{r-1}$$

$$= \frac{300[(1+\frac{0,05}{12})^{36} - 1]}{(1+\frac{0,05}{12}) - 1}$$

$$= \frac{300[(1+\frac{0,05}{12})^{36} - 1]}{\frac{0,05}{12}}$$

$= \text{R}11\ 626$ *Compare with the formula in 4 on page 29.

annuity:
an annuity is a sequence of regular, equal payments where interest is either accrued or charged

amortise:
a loan is said to be amortised when it, together with all interest charged, is paid off by means of an annuity

scrap value:
the final value of an asset the day it is disposed of (scrapped) after depreciating

sinking fund:
an annuity which is set up to provide the money to purchase a product some time in the future.

b) This amount receives interest for another month to reach the end of 3 years (36 months), so using the normal C.I. formula

$$A = 11\ 626(1+\frac{0,05}{12})^1 = R11\ 674,44$$

1. The formula for the geometric series gave us the amount **immediately** after the final payment (before any further interest accrued).
2. This is an example of an annuity because a fixed amount of R300 was paid into the account every month (regular, equal amounts).
3. We calculated the Future Value of the annuity (i.e. the value after 36 payments).
4. If an annuity is made up of n equal payments of Rx at an interest rate of i per period then the Future Value **immediately** after the nth payment is:

   $$F_V = \frac{x[(1+i)^n - 1]}{i}$$ Compare with the formula in a).

   In the above example $x = 300,\ i = \frac{0,05}{12}$ and $n = 36$.
5. If the money is left in the account after the last payment then interest must be added for the additional period.

ACTIVITY 6

A teacher makes regular payments, at the beginning of each month, of R250 into a fund offering 6% p.a. compounded monthly. If the interest rate does not change how much will she receive:

a) at the end of 20 years?
b) at the end of 30 years if she stops paying into the fund after 20 years?

ANSWERS ON PAGE 82

Present Value of an Annuity, P_V:

If we want to borrow money (perhaps take out a bond on a house) which will then be amortised (paid back, including all interest, in equal amounts on a regular basis) then we need to calculate how much we can borrow now. This is the Present Value of the annuity.

Example:
How much money can Nunu borrow now (i.e. what is the present value of the annuity) if she intends to pay R1 500 per month for 12 months and the bank charges 15% p.a. compounded monthly.

Solution:

The amount she can borrow now is P_V, the Present Value of the annuity.

The Future Value of the annuity is $F_V = \frac{x[(l+i)^n - 1]}{i}$ where $x = 1500,\ i = \frac{0,15}{12},\ n = 12$

But the Future Value of the annuity is also $F_V = P_V(1+i)^n$ (i.e. the present value plus compound interest).

So

$$P_V = (1+\frac{0,15}{12})^{12} = \frac{1\ 500[(1+\frac{0,15}{12})^{12} - 1]}{\frac{0,15}{12}}$$

$$\therefore P_V = \frac{1\ 500.[(1+\frac{0,15}{12})^{12} - 1].(1+\frac{0,15}{12})^{-12}}{\frac{0,15}{12}}$$

$$= \boxed{\frac{1\ 500[1-(1+\frac{0,15}{12})^{-12}]}{\frac{0,15}{12}}}$$

* Compare with the formula in a) (page 28) and 4 (page 29).

$$= R16\ 618,97$$

Nunu can borrow R16 618,97 now, but because of the interest she will have to pay back R1 500 x 12 = R18 000 (i.e. R1 381,03 interest).

The assumption is that she only starts paying back the loan 1 month after she takes the money (it would not make sense to make the first repayment on the day she receives the loan).

By solving $P_V(1+i)^n = \frac{x[(1+i)^n - 1]}{i} = F_V$

you can show $\boxed{P_V = \frac{x[1-(1+i)^{-n}]}{i}}$ Compare with the formula above.

ACTIVITY 7

1. A house bond is amortised by 240 monthly payments (i.e. over 20 years) of R3 000. Determine the amount of the bond (present value of the annuity) if interest is charged at 12,5% p.a. compounded monthly.

2. How much must be paid every month to amortise a loan of R15 000 over 10 months if interest is charged at 15% p.a. compounded monthly?

3. How much must you ask your rich uncle to deposit into your bank account (which earns 4,5% p.a. compounded monthly) on the 1st December this year in order to enable you to draw R1 000 pocket money, on the 1st day of each month next year, when you are at college?

4. The International Lotto offers a prize worth R5 000 000 paid out as follows: R1 000 000 immediately and R160 000 each year for 25 years starting in a year's time. The Lotto company can purchase an annuity giving 10% p.a. compounded daily.

   a) What is the effective annual rate of the annuity?
   b) How much money does the Lotto company need in total to pay out the R1 000 000 and invest in the annuity on the day of the draw?

5. Derive the formula $P_V = \frac{x[1-(1+i)^{-n}]}{i}$ for the present value of an annuity paying Rx per year for n years at an interest rate if i p.a.

6. Sometimes it is important t o know the 'outstanding balance' on a loan after a certain number of payments. The outstanding balance on a loan (amount still owing) is the present value of all the remaining payments. A loan of R10 000 at 15% p.a. compounded monthly is amortised by 24 equal monthly payments.

   a) Calculate the monthly instalments
   b) Calculate the outstanding balance on the loan immediately after the 10th instalment has been paid.

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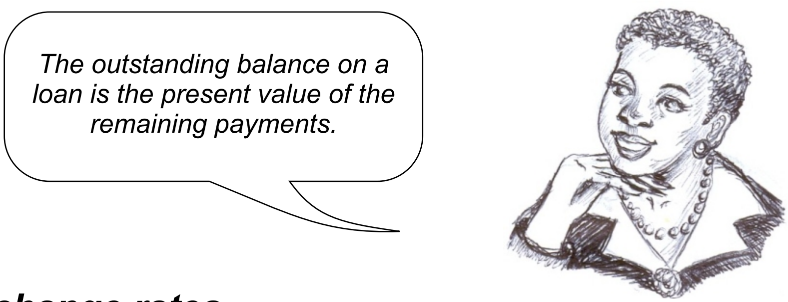

Exchange rates

When we buy goods from a foreign country, or visit a foreign country, we normally have to pay in their currency. The relationship between currencies is known as the exchange rate. Exchange rates re available from newspapers, radio, T.V., banks, exchange bureaus and the internet.

Example:
The exchange rates on a particular day are given over the radio as follows:

1 USD (United States Dollar)	=	R7,09
1 Euro	=	R10,05
1 British Pound	=	R11,42
1 Botswana Pula	=	R1,04

This means that on that day goods costing \$100 (100 USD) will cost $100 \times 7{,}09$ = R709, and if you exchange rand for euros the bank will charge you $10{,}05 \times 500$ = R5 025 for €500 (500 Euros) (in practice the bank will also add taxes and other charges!), and if you take R1 000 to an exchange bureau you will get $\frac{100}{1{,}04} = 961{,}54$ Botswana Pula (probably less because, once again, there will be additional charges by the bureau).

If the next day the exchange rate increases, for example to R7,20 for \$1, then we say the rand has **weakened** against the dollar (you will have to pay more for \$1).

On the other hand, if the exchange rate decreases, for example to R9,50 for €1, then we say the rand has **strengthen** against the euro (you will have to pay less for €1). It is quite possible for the rand to weaken against one currency but strengthen against another on the same day.

The following table of foreign exchange rates was obtained from a local South African bank on the 1st September 2011.

CURRENCY	SELLING	BUYING
U.S. DOLLAR ($)	7.3100	6.9600
EURO (€)	10.2909	9.7677
BRITISH POUND (£)	11.7746	11.1673
JAPANESE YEN (¥)	10.4144	11.1132
AUSTRALIAN DOLLAR	0.1296	0.1368
BOTSWANA PULA	0.8695	1.0195
INDIAN RUPEES	6.2484	6.6780
KENYAN SHILLING	12.5000	14.0000
MALAWI KWACHA	22.1500	23.6500

- **Selling** rates are applicable to the conversion of Rand into foreign currency i.e. the rates at which the bank will sell you the foreign currency. For example, the bank will sell \$1 to you for R7.31. This is the rate that the bank would use when you want foreign currency to travel to a foreign country.
- **Buying** rates are applicable to the conversion of foreign currency or travellers cheques into Rand.
- For example, the bank will buy €1 from you for R9.77. This is the rate the bank would use when you return from a foreign country and want to convert your foreign currency back into Rand.
- In this table US Dollars, UK Pound Sterling and the Euro are quoted as the number of Rand per one unit of currency, as in the earlier examples.

- All other currencies are quoted the other way around, as the number of foreign currency units per one Rand. So, for example, Rl will buy (i.e. the bank will sell you) 12.50 Kenyan Shillings. In other words, the bank will sell you 1 Kenyan shilling for

  $\frac{1}{12,50} = 0,08$ Rand (i.e. 8 South African cents).

- The foreign exchange rates given on the radio or TV are the rates set by the SARS (South African Revenue Services, fondly known as 'the Tax Man') and are usually close to the average of the buying and selling rates quoted by the banks and exchange bureaus.

ACTIVITY 8

Use the table of foreign exchange rates on the previous page to answer the questions that follow.

1. You have just returned from a trip overseas and you take the following foreign currency to the bank to exchange for Rand: €1 520, £725 and ¥16 000.

   a. Which column of the table is applicable, **buying** or **selling**?
   b. How many Rand will you get for ¥1 (i.e. for 1 Japanese Yen)?
   c. How many Rand will the bank give you, assuming no extra charges?

2. You intend going on a Safari to Botswana, Malawi and Kenya and go to the bank to exchange Rand into foreign currency. You decide that you would like R2 000 worth of each of the countries' foreign currencies and another $1 000 for emergencies.

   a) Which column of the table is applicable, **buying** or **selling**?
   b) How many South African Rand will the bank charge you, in total, for tile currency if they charge 1 % commission?
   c) How many Pula, Kwacha and Kenyan Shillings will you get from the bank?

3. Sipho sells the bank 1 000 Australian dollars in exchange for a different currency.

   a) How many Indian Rupees would he get for his Australian Dollars?
   b) How many Euros would he get for his Australian Dollars?
   c) Later that day he takes the Euros back to the bank and changes them back to Australian Dollars. How many Australian Dollars will he get from the bank? How much money has he lost in Rands?

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Summary

- Simple interest is calculated on the initial investment only.
- The formula for simple growth/decay is $A = F_v = P_v\,(1 \pm ni)$ (straight line graph).
- Compound interest is calculated on the initial investment plus all interest already earned (exponential graph).

  The formula for compound growth/decay is $A = F_v = P_v\ (1 \pm i)^n = P_v\,(1 \pm \quad)^{nt}$ where t is the conversion factor from interest period to the compounding period.
- The difference between nominal and effective rates.

  If i is the nominal rate per annum and j the effective annual rate then $1 + j = (1 + \frac{i}{t})^{nt.}$
- The meanings of the terms annuity, amortise, scrap value and sinking fund.
- The future value of an annuity **immediately** after the n^{th} payment of Rx at an interest rate of i is

$$F_v = \frac{x[(1+i)^n - 1}{i}$$

- The present value of an annuity **one payment period before** the first of n payments of Rx at an interest rate of i is

$$P_v = \frac{x[1-(1+i)^{-n}]}{i}$$

- The outstanding balance of a loan is the present value of the remaining payments.
- Foreign currency exchange rates change continuously relative to other currencies weakening and strengthening, and foreign exchange (FOREX) outlets usually sell and buy a particular currency at different rates.

CHECKLIST

Are you able to:

- calculate simple and compound interest
- identify the difference between nominal and effective rates
- understand the meaning of annuity, amortise, scrap value and sinking funds
- work out the present and future value of an annuity
- work with foreign exchange rates

SELF-CHECK EXERCISE

1. How much money will you have after 5 years if you invest Rl 200 at 4,5% p.a.

   a) simple interest?
   b) compounded annually?
   c) compounded every three months?
   d) compounded daily?

2. Calculate the effective annual rates of each of the investments in 1. above.

3. You start investing R100 per month on the 1st January 2012. How much money will you have at the end of December 2016 if interest is 4% p.a. compounded monthly?

4. You can afford to pay R3 500 at the end of each month for a bond on a flat you wish to purchase.

   a) How much can you borrow at a rate of 10,5% p.a. compounded monthly if the bond must be paid off in 20 years?
   b) What is the outstanding balance on the loan after 10 years?
   c) What fraction of the bond has been paid off? Explain.

5. Use the table of foreign exchange rates given before Activity 8 to answer this question.

   a) After a trip overseas you exchange £270 and ¥1 200 for Rand. How much do you receive?
   b) 5 days later, the Rand has weakened against the U.S. Dollar by 12c but strengthened against the Pula by 5c. How many Dollars and how many Pula will you get if you purchase Rl 000 of each currency?
   c) You purchase R10 000 worth of Indian Rupees on the lst September 2011. On a trip to India you spend 50 000 Rupees. When you return the Rand has weakened against the Rupee by 5c. How many Rand will you get when you exchange your remaining Rupees?

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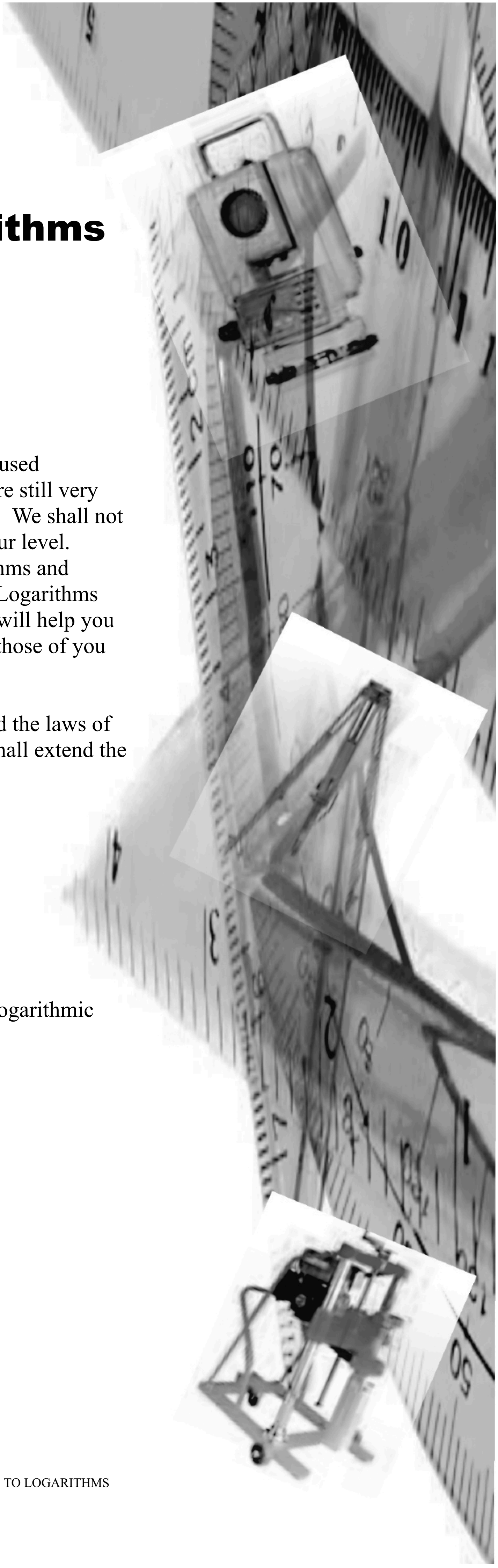

LESSON 4

From exponents to logarithms

About this lesson

Before the invention of the electronic calculator, people used logarithms to solve complex calculations. Logarithms are still very useful in Advanced Mathematics, especially in Calculus. We shall not consider this application in Calculus since it is above your level. Instead we shall look at the relationship between logarithms and exponents and learn some basic properties about them. Logarithms link closely with exponents and a basic knowledge of it will help you to understand exponents well. This work will also help those of you who intend doing Mathematics at a higher level.

You learned about exponents previously. You considered the laws of exponents in the domain of integers. In this lesson we shall extend the domain to cover rational numbers.

You will also learn to solve simple exponential equations.

In this lesson you will:

- work with integers and rational numbers
- calculate equations involving exponents
- understand the laws of logarithms and work with logarithmic equations
- change the base

Exponents

You should know that:

$$a^{-m} = \frac{1}{a^m} \text{ and } a^0 = 1 \ (a \neq 0)$$

$$a^m \times a^n = a^{m+n}$$

$$a^m \div a^n = a^{m-n}$$

$$(a \times b)^n = a^n \times b^n$$

$$(a^n)^m = a^{n \times m}$$

These are called the laws of exponents.

These laws are also true if m and n are rational numbers and $a > 0$, $b > 0$.
For example:

if $m = \frac{1}{3}$ and $n = \frac{2}{3}$, then

$$a^{\frac{1}{3}} \times a^{\frac{2}{3}} = a^{\frac{1}{3}+\frac{2}{3}} = a$$

$$a^{\frac{2}{3}} \div a^{\frac{1}{3}} = a^{\frac{2}{3}-\frac{1}{3}} = a^{\frac{1}{3}}$$

$$(a \times b)^{\frac{2}{3}} = a^{\frac{2}{3}} \times b^{\frac{2}{3}}$$

$$(a^{\frac{2}{3}})^{\frac{1}{3}} = a^{\frac{2}{3} \times \frac{1}{3}} = a^{\frac{2}{9}}$$

$$a^{-\frac{1}{3}} = \frac{1}{a^{\frac{1}{3}}} \ (a \neq 0)$$

Now what do we mean by $a^{\frac{1}{2}}$? Try the following on your calculator:

a) $25^{\frac{1}{2}}$ b) $100^{\frac{1}{2}}$

You should have got the following answers.

a) $25^{\frac{1}{2}} = 5$ b) $100^{\frac{1}{2}} = 10$

In each case the answer is the square root of the base.

$a^{\frac{1}{2}} =$ square root of $a = \sqrt{a}$

Try the following on your calculator:

$8^{\frac{1}{3}}$; $27^{\frac{1}{3}}$; and $64^{\frac{1}{3}}$

You saw that $8^{\frac{1}{3}} = 2$; $27^{\frac{1}{3}} = 3$; and $64^{\frac{1}{3}} = 4$.

This means that $y = (\frac{1}{2})^x$

In general $a^{\frac{1}{n}} = n^{th}$ root of $a = \sqrt[n]{a}$ where n is a positive integer > 1 and $a > 0$.

ACTIVITY 1

1. Without using a calculator, work out the following:

a) $81^{\frac{1}{2}}$ b) $64^{\frac{1}{3}}$

c) $81^{\frac{1}{4}}$ d) $16^{\frac{1}{4}}$

2. Try and work out the following without a calculator:

a) $8^{-\frac{1}{3}}$ b) $9^{-\frac{1}{2}}$

c) $32^{-\frac{1}{5}}$ d) $64^{-\frac{1}{3}}$

Instead of writing $\sqrt[2]{a}$, for the square root of a, we usually write $\sqrt{a}$. That is, we leave out the 2 in the square root sign.

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Also, $\sqrt[n]{0} = 0$ for any positive integer n 1.

However, to avoid exceptions we define $a^{\frac{1}{n}} = \sqrt[n]{a}$ only for $a > 0$.

It is possible to have $a < 0$ if n is an odd positive integer e.g. $\sqrt[3]{-8} = -2$

Equations involving exponents

Do you still remember the terms **base** and **exponent**? For example, the number 8^3 has a base 8 and an exponent 3. If two numbers have the same base, then the two numbers are equal if they have the same exponent as well. For example if $2^3 = 2^x$, then $x = 3$. We use this idea to solve equations involving exponents.

To solve equations involving exponents, first re-write the numbers on both sides of the equal sign (=) to have the same base. Then equate the exponents and solve for the unknown.

Examples

a) $2^x = 32$ b) $125^x = \frac{1}{25}$ c) $2^{2x-1} = 4^{-x}$

Solutions

a) We know that $32 = 2^5$ therefore

$2^x = 32$ gives

$2^x = 2^5$ which makes

$x = 5$ because the base on both sides of the equation is 2.

b) $125^x = \frac{1}{25}$

Here we must rewrite both sides so that they have the same base.

$(5^3)^x = \frac{1}{5^2}$

$\therefore 5^{3x} = 5^{-2}$

Therefore this gives $3x = -2$

so $x = -\frac{2}{3}$

c) $$2^{2x-1} = 4^{-x}$$

$$2^{2x-1} = (2^2)^{-x}$$

$$2^{2x-1} = 2^{-2x}$$

this gives $2x - 1 = -2x$

which gives $2x + 2x = 1$

$$4x = 1$$

$$x = \frac{1}{4}$$

ACTIVITY 2

Solve for x in the following:

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a) $2^{x-1} = 1$

b) $3^x = \frac{1}{27}$

c) $2^{x+1} = (0,5)^{x-2}$

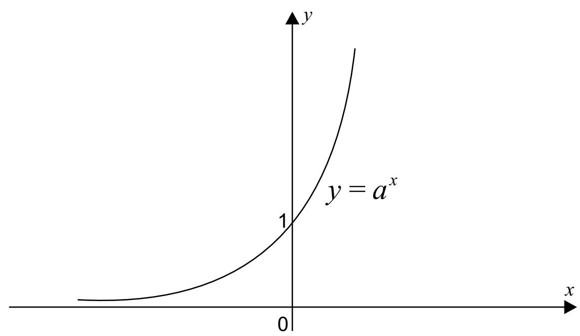
d) $27^{x-2} = 81^{2x+1}$

The graph of the exponential function

$y = a^x, a > 0, x \neq 1$

A function of the form $y = a^x, a > 0$ and $a \neq 1$ with x being a real number is known as an exponential function with base a. The size and direction of the graph of the exponential function depends on the value of a.

If $a > 1$ the graph of $y = a^x$ looks like this:



The graph has the following properties:

1. The y –intercept is always 1. You should know that the y –intercept of any graph is obtained by putting $x = 0$ in the equation. Also any number (except 0), raised to the power zero is 1. So put $x = 0$ in the function $y = a^x$. This gives $y = a^0 = 1$.

2. When $x = 1$, $y = a$. In other words the point (1;a) always lies on the graph of $y = a^x$. Put $x = 1$ in the function $y = a^x$ to verify this yourself.

3. As x increases, y increases. To show this let's take a to be 2. When $x = 1$, y is 2 and when $x = 2$, y is 4. Each time we increase the value of x, the value of y also increases. This means $y = a^x$, for $a > 1$ is an increasing function.

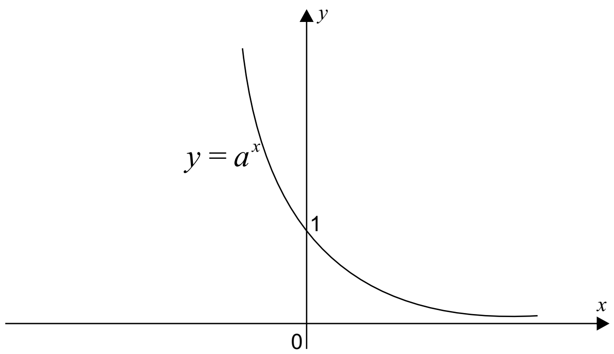
4. The domain of the graph is the set of real numbers. This means that x can be any real number.

5. The range is the set of positive real number (i.e. y is always positive). This is because even when x takes a negative value, y will be a fraction which is still positive. Therefore the curve lies above the x –axis.

6. As x decreases the curve approaches the x –axis but it never touches it. We say the x –axis is an **asymptote**. The x –axis is the line $y = 0$ so let's take $a = 2$ and show that as we decrease the value of x we will be getting closer and closer to 0.

x	4	3	2	1	0	–1	–2	–3	–4	–5
$y = 2^x$	16	8	4	2	1	0,5	0,25	0,125	0,0625	0,003125

If $0 < a < 1$, the graph of $y = a^x$ looks like this:



It has the following properties:

1. The y –intercept is always 1.

2. When $x = 1$, $y = a$. In short, the point $(1;a)$ always lies on the graph $y = a^x$.

3. As x increases, y decreases. In other words, $y = a^x$ is a decreasing function if $0 < a < 1$.

4. The domain is the set of all real numbers.

5. The range is the set of positive real numbers.

6. As x increases, the curve approaches the x –axis but it never touches it. This means the x –axis is asymptotic to the graph $y = a^x$.

Verify these properties by going through the same procedure as in the first case.

ACTIVITY 3

1. a) Use your calculator to complete the table below for the function $y = 3^x$.

x	–5	–4	–3	–2	–1	0	1	2
y	0,0041							9

b) Use your table in a) to draw the graph of the exponential function $y = 3^x$.

2. a) Complete the table below for the function $y = \left(\frac{1}{2}\right)^x$.

x	–3	–2	–1	0	1	2	3	4	5
y	8			1					

b) Use this table to draw the graph of $y = \left(\frac{3}{2}\right)^x = 1,5^x$

3. Make rough sketches of the following graphs:

a) $y = \left(\frac{1}{3}\right)^x = \frac{1}{3^x}$

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b) $y = \left(\frac{3}{2}\right)^x = 1,5^x$

The graph of $y = a \bullet k^{x} + q, k > 0, k \neq 1$

You have studied the effects of a, p and q on hyperbolas, parabolas and trigonometric functions of the form

$$y = \frac{1}{x+q},\ y = \frac{a}{x},\ y = a(x+p)^2 + q,\ y = a\sin(x-p)+q,\ \text{etc.}$$

The effects on the graph of $y = k^x$ are the same.

ACTIVITY 4

1. Sketch each of the following graphs on the same set of axes. Show the horizontal asymptotes.

a) $y = 2^x$ b) $y = 2^x + 1$ c) $y = 2^x + 2$ d) $y = 2^x - 1$

2. Sketch each of the following graphs on the same set of axes.

a) $y = 3^x$ b) $y = -(3^x) = -3^x$ c) $y = 2 \cdot \frac{1}{2}^{x}$

3. Sketch each of the following graphs on the same set of axes.

a) $y = \left(\frac{1}{2}\right)^x$ b) $y = 2 \cdot \left(\frac{1}{2}\right)^x$ c) $y = 3\left(\frac{1}{2}\right)^x$

4. Sketch each of the following graphs on different sets of axes.

a) $y = 2^{x+1} - 3$ b) $y = -\left(\frac{1}{3}\right)^x + 2$

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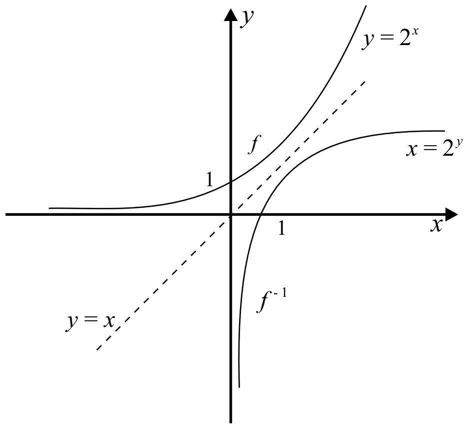
The logarithmic function and logarithms

Remember that the reflection of $y = f(x)$ about the line $y = x$ gives the inverse function of *f*.

Example:

$y = f(x) = 2^x$



None of our known techniques or functions will enable us to write with y as the subject of the formula i.e. in the form $y =$... . We solve this by defining a new function, called a logarithmic function.

Definition:
$y = \log_a x$ if $x = a^y$ $(a > 0,\ a \neq 1)$

Since $a > 0, x = a^y > 0$. $\log_a x$ is not defined for ≤ £ 0.

Example:

a) If $x = 2^y$ then $y = \log_2 x$. (log of x base 2)

b) If $y = \log_{10} x$ then $x = 10^x$

c) Since $2^3 = 8$, $\log_2 8 = 3$

d) $\log_3 81 = 4$ because $3^4 = 81$

e) $10^{-3} = \frac{1}{1000}$ $\therefore$ $\log_{10} \frac{1}{1000} = -3$

Note that before the days of scientific calculators tables of calculated logarithms were used ('log tables') and were generally compiled for base 10.

These became known as **common logarithms** and it was customary to omit the base when writing the logarithm. Thus log2 meant $\log_{10}2$. Many books and calculators still use this convention. Your calculator probably has three logarithm buttons: log, In and log ▪▯

- The button log means $\log_{10}$
- The button In means loge, where e = 2,71 828... This logarithm is called the natural logarithm and is very important in advanced mathematics.
- The button log▪▯ allows you to choose your own base ▪ (▪>0 and ▪≠1). Experiment with your calculator.

ACTIVITY 5

1. Write the following in logarithmic form:

a) $5^3 = 125$ b) $64 = 8^2$

c) $3^{-1} = \frac{1}{3}$ d) $4 = 8^{\frac{2}{3}}$

2. Express in exponential form:

a) $\log_{0,1} 100 = -2$ b) $\log 10 = 1$

c) $\log_a b = c$ d) $\log_2 16 = 4$

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We can use the idea of the relationship between logarithms and exponents to find the value of logarithms.

Example

Find the value of $\log_2 8$

Solution

Let $y = \log_2 8$

Then $2^y = 8 = 2^3$

Therefore $y = 3$ or $\log_2 8 = 3$

ACTIVITY 6

1. Find the values of:

a) $\log_2 \frac{1}{16}$ b) $\log 100$ c) $\log_7 49$

d) $\log_a a (a > 0,\ a \neq 1)$ e) $\log_a 1 (a > 0,\ a \neq 1)$

2. Use your calculator to find

a) $\log 2$ b) $\ln 2$ c) $\log_4 2$ d) $\log_{\frac{1}{2}} 2$

3. On the same set of axes draw sketch graphs of

a) $y = 3^x$ b) $y = \log_3 x$ c) $y = (\frac{1}{3})^x$ d) $y = \log_{\frac{1}{3}} x$

4. Draw sketch graphs of the following functions on the same set of axes

a) $y = \log_a x$ for $a > 1$ b) $y = \log_a x$ for $0 < a < 1$

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The laws of logarithms

Do you still remember the laws of exponents? If not, go back to the beginning of the lesson and revise them. We shall use the laws of exponents to prove the laws of logarithms.

The basic laws of logs (short for logarithms) for $a > 0, a \neq 1, b > 0, c > 0$.

1. $\log_a b^m = m\log_a b$
2. $\log_a (bc) = \log_a b + \log_a c$
3. $\log_a \frac{b}{c} = \log_a b - \log_a c$

These laws only hold if the base is positive and not equal to 1. We also often use: $\log_a a = 1$ and $\log 1 = 0$.

The proofs of the laws

Law 1

$\log_a b^m = m\log_a b$

Let's define $x = \log_a b$

We therefore have $a^x = b$

So if we raise both sides to the power m we get $(a^x)^m = b^m$

therefore $\log_a b^m = xm = mx$ ①

but we have defined $x = \log_a b$, if we now replace x with $\log_a b$ in ①

we get $\log_a b^m = m\log_a b$ **(law 1)**

Example

Find $\log_8 (64)^2$

Solution

$$\begin{aligned}\log_8 (64)^2 &= \log_8 (8^2)^2 \\ &= \log_8 8^4 \\ &= 4\log_8 8 \\ &= 4.1 \\ &= 4\end{aligned}$$

Law 2

> *Note that* $\log_a b^m$ *is not the same as* $(\log_a b)^m$*.*
>
> *For* $\log_a b^m$ *you first raise* ***b*** *to the power* ***m*** *then take the log base* ***a****, but for* $(\log_a b)^m$ *you first take the log base* ***a*** *of* ***b*** *then raise to the power* ***m****.*

$\log_a bc = \log_a b + \log_a c$

Let $x = \log_a b$ and $y = \log_a c$

Then $b = a^x$ and $c = a^y$

so $bc = (a^x)(a^y) = a^{x+y}$ (using the first law of exponents)

i.e. $\log_a(bc) = x + y$

but $x = \log_a b$

and $y = \log_a c$

therefore $\log_a bc = \log_a b + \log_a c$ **(law 2)**

Law 3

$$\log_a \frac{b}{c} = \log_a(b.c^{-1})$$
$$= \log_a b + \log_b c^{-1} \quad \text{(law 2)}$$
$$= \log_a b - \log_b c \quad \text{(law 1)}$$

Example

Find $\log_2(8 \times 4)$.

Solution

$$\log_2(8 \times 4) = \log_2(2^3 \times 2^2)$$
$$= \log_2 2^3 + \log_2 2^2$$
$$= 3\log_2 2 + 2\log_2 2$$
$$= 3 + 2$$
$$= 5$$

Example

Find $\log_5\left(\frac{625}{25}\right)$

Solution

$$\log_5\left(\frac{625}{25}\right)$$
$$= \log_5 625 - \log_5 25$$
$$= \log_5 5^4 - \log 5^2$$
$$= 4\log^5 - 2\log_5 5^5$$
$$= 4 - 2$$
$$= 2$$

Example:
Evaluate $\log 20 + \log 3 - \log 6$.

Solution:

$\log 20 + \log 3 - \log 6$ **from laws 1 and 2**

$= \log \frac{20x3}{6}$

$= \log \frac{60}{6}$

$= \log 10$ (base assumed to be 10)

$= 1$

Before you do any activities, look at some common mistakes that learners tend to make.

Mistake	Correction	Explanation
$\log 20$ $= \log(10+10)$ $= \log 10 + \log 10$ $= 1+1$ $= 2$	$\log 20$ $= \log(2 \text{ x } 10)$ $= \log 2 + \log 10$ $= \log 2 + 1$	$\log(10+10) \neq \log 10 + \log 10$ Law 1 says: $\log(10 \text{ x } 10) = \log 10 + \log 10$
$\frac{\log 64}{\log 4}$ $= \frac{\log 16}{\log 1}$ $= \log 16$	$\frac{\log 64}{\log 4}$ $= \frac{\log 4^3}{\log 4}$ $= \frac{3 \log 4}{\log 4}$ $= \frac{3 \log 4}{\log 4}$ $= 3$	You cannot cancel in logarithmic form unless you are cancelling the whole log. For example: $\frac{\log 4}{\log 4} = 1$
$\frac{\log 64}{\log 4}$ $= \log 64 - \log 4$	As above	Law 3 says that: $\log \frac{64}{4}$ $= \log 64 - \log 4$
$\frac{\log 25 - \log 5}{\log 25 + \log 5}$ $= \frac{1-1}{1+1}$ $= 0$	$\frac{\log 25 - \log 5}{\log 25 + \log 5}$ $= \frac{\log 5^2 - \log 5}{\log 5^2 + \log 5}$ $= \frac{2 \log 5 - \log 5}{2 \log 5 + \log 5}$ $= \frac{\log 5(2-1)}{\log 5(2+1)}$ $= \frac{1}{3}$	Because logs are numbers, they obey the rules of algebra. You cannot cancel across an addition or subtraction sign.

ACTIVITY 7

1. Simplify the following:

   a) $2\log 3 + \log 2 - \log 4$
   b) $2\log_a 5 + \log_a 4 + 2\log_a 10$
   c) $2 - 2\log 5$

2. Expand the following:

   a) $\log \dfrac{abc}{d}$

   b) $\log(6 \text{ x } 4)$

   c) $\log \sqrt{\dfrac{a}{b}}$

3. Evaluate:

   a) $\log_2 32 - \log_2 8$

   b) $\log 8 + \log 125 - 1$

   c) $\dfrac{\log 16 - \log 4}{\log 16 + \log 4}$

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Logarithmic equations

An equation of the form $k = \log_a x$ is known as a logarithmic equation. You can solve an equation like this by using one of two methods. The first method is to convert it to an exponential equation. The second method uses the fact that if $\log_a x = \log_a y$ then $x = y$.

Example
Solve for x: $2\log x = 8$

Solution

Method 1

$$2\log x = 8$$
$$\log x = 4 = 4\log_{10} 10$$
$$\log x = \log 10^4$$
$$x = 10^4 = 10\,000$$

Method 2

$$2\log x = 8$$
$$\log x = 4 = 4\log_{10} 10$$
$$\therefore \log x = \log 10^4$$
$$x = 10^4 = 10\,000$$

Example

Solve for x if $\log_3 5 + \log_3 x = 1$

Solution

Method 1

$\log_3 5 + \log_3 x = 1$

gives $\log_3 5x = 1$

$\log_3 5x = \log_3 3$

therefore $5x = 3$

$$x = \frac{3}{5}$$

Method 2

$\log_3 5 + \log_3 x = 1$

$\log_3 5x = 1$

$\therefore 5x = 3^1$

$$x = \frac{3}{5}$$

ACTIVITY 8

Solve for x:

a) $\log_a x + \log_a 6 = \log_a (x+6) \quad (a > 0, a \neq 1)$
b) $3\log_2 x - 3 = 0$
c) $2\log x = 4$

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Summary

In this lesson you have studied and used the following ideas and results if m and n are rational numbers and $a > 0$, $b > 0$.

$a^{-m} = \frac{1}{a^m}$ and $a^0 = 1$

$a^m \times a^n = a^{m+n}$

$a^m \cdot a^n = a^{m-n}$

$(a \times b)^n = a^n x b^n$

$(a^n)^m = a^{nm}$

Two numbers in exponential form are equal if they have the same base and exponent. For example, if $2^3 = 2^x$ then $x = 3$. We use this idea to solve exponential equations.

$\log_a x = y$ is equivalent to $x = a^y$ and $a > 0$, $a \neq 1$

The laws of logarithms for $a > 0, a \neq 1, b > 0, c > 0$ are:

$\log_a b^m = m \log_a b$

$\log_a bc = \log_a b + \log_a c$

$\log_a \frac{b}{c} = \log_a b - \log_a c$

CHECKLIST

Are you able to:

- solve equations by two different methods. The first is to change it to exponential equation $a^k = x$. The second method uses the fact that if $\log_a x = \log_a y$ then $x = y$.
- draw graphs of the exponential functions by plotting a few points.
- understand the general properties of the exponential graph.
- draw the graph of the logarithmic functions by reflecting the equivalent exponential function in the line $y = x$.
- draw the graph of the logarithmic function by plotting a few points.

SELF-CHECK EXERCISE

1. Solve for x in the following:

   a) $2x^{\frac{1}{2}} = 4$ b) $27^x \times 9^{x-2} = 1$

   c) $2^{2x-1} = 8$ d) $4^x = 64$

2. Write the following in exponential form:

   a) $\log_4 x = 3$ b) $\log_3 243 = 5$

   c) $\log 0,0001 = y$ d) $\log_8 4 = \dfrac{2}{3}$

3. Simplify:

   a) $\log 3 + \log 4$ b) $\log 6 - \log 2$

   c) $\log_a 2 + \log_a 6 - \log_a 4$ d) $3\log xy - 2\log x - 2\log y$

4. Solve for x in the following:

   a) $\log(x+4) = 1$ b) $\log_x 243 = 5$

   c) $\log(3x+21) - \log(2x+1) = \log 8$ d) $\log_3(x-4) + \log_3(x-2) = 1$

5. Sketch each of the following pairs of graphs on the same set of axes.

   a) $y = 2^x$, $y = \log_2 x$

   b) $y = \left(\dfrac{1}{3}\right)^x$, $y = \log_{\frac{1}{3}} x$

6. Find the values of $y = \log_4 x$ for $x = \dfrac{1}{16}, \dfrac{1}{4}, 1, 4, 16$ and use them to draw the graph of $y = \log_4 x$.

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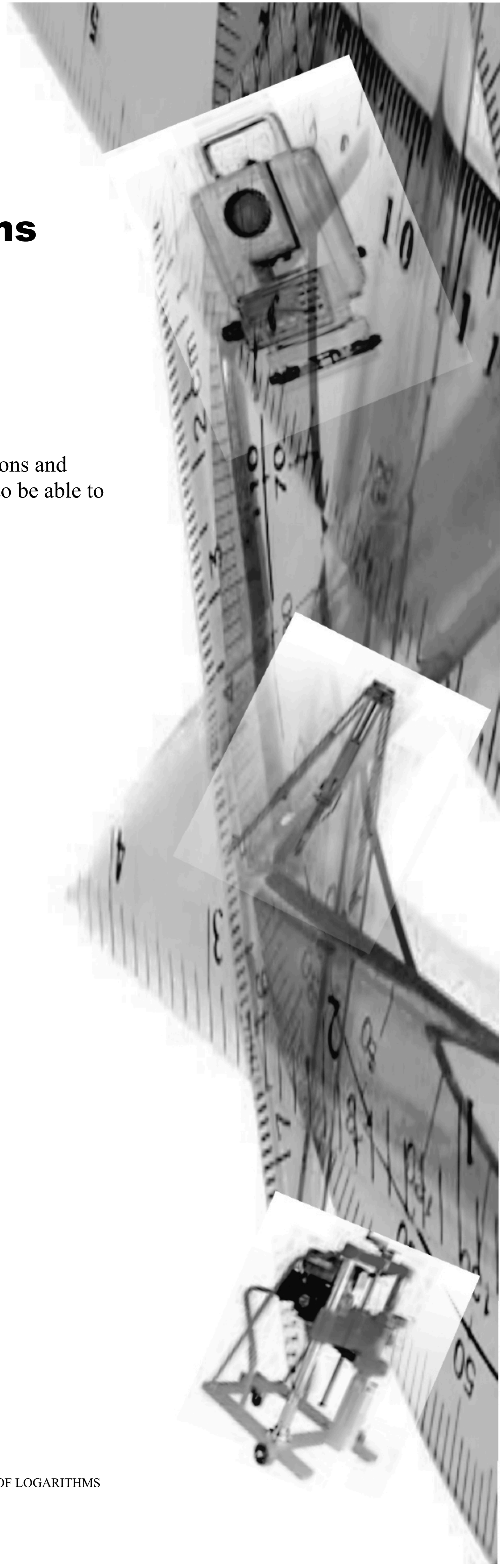
LESSON 5

Applications of logarithms



About this lesson

In this lesson we are going to look at exponential equations and financial mathematics using logarithms. It is important to be able to solve exponential equations using logarithms.

In this lesson you will:

- work with logarithms
- calculate fixed payment annuities
- work with fluctuating interest rates
- solve exponential equations
- calculate compound interest

Exponential equations using logarithms

So far we have only been able to solve exponential equations of the type $a^x = b$ where b has been a power of a.

Example

We can solve $2^x = 8$ because $8 = 2^3$ so $2^x = 2^3$ and hence $x = 3$.
But how do we solve $2^x = 7$?

The answer is to take the logarithm of both sides (any base will work but we shall use 10).

$$\begin{aligned} 2^x &= 7 \\ \log 2^x &= \log 7 \\ x\log 2 &= \log 7 \\ x &= \frac{\log 7}{\log 2} \\ &= \frac{0,8450...}{0,3010...} \quad \text{(by calculator)} \\ &= 2,807 \quad \text{(correct to 3 decimals)} \end{aligned}$$

ACTIVITY 1

Solve the following exponential equations using logarithms. Give answers correct to 2 decimal places.

1. $3^x = 2$
2. $5^{-x} = 12$
3. $\left(\frac{1}{2}\right)^{x+1} = 5$
4. $(1,12)^n = 1,315$
5. $\frac{260[1-1,02^{-n}]}{0,02} = 3600$

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Logarithms and financial mathematics

This technique is very useful for calculating the period n in our financial mathematics formulas.

Example

R1 000 invested at 4% p.a. compounded annually amounts to R1 216,65. Calculate the period of the investment.

Solution

$$\begin{aligned} F_v &= P_v(1+i)^n \\ 1216,65 &= 1000(1+0,04)^n \\ 1,04^n &= \frac{1216,65}{1000} = 1,21665 \\ \log 1,04^n &= \log 1,21665 \\ n\log 1,04 &= \log 1,21665 \\ n &= \frac{\log 1,21665}{\log 1,04} = 4,9999... \text{ years} \\ \text{i.e. } n &= 5 \text{ years} \end{aligned}$$

If you are asked for years and days, always assume 365 days in a year.

$$8{,}31 \text{ years} = 8 \text{ years and } \frac{31}{100} \times 365$$
$$= 8 \text{ years, } 113 \text{ days}$$

ACTIVITY 2

1. R500 is invested at 6% p.a. compound interest compounded monthly. How long will it take for the money to double? Express your answer in years and days. Assume there are 365 days in a year.

2. Monthly payments of R500 are made into an account paying 3,5% p.a. compounded monthly. After how many months will the annuity reach R50 000?

3. A company has a policy that it will sell a vehicle when it has depreciated to 15% of its original value. If depreciation is calculated on the reducing-balance basis at 20% p.a., calculate after how many years the vehicle will be sold.

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Miscellaneous calculations in financial mathematics

The following examples are for enrichment and may be omitted if you are pressed for time. They are however representative of what really does happen in real life.

Fixed payment annuity

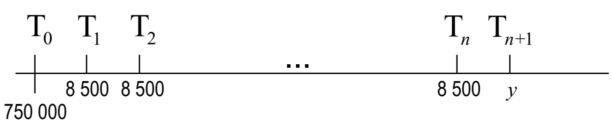
Usually, we can only afford to pay a fixed amount every month to amortise a loan. In a case like this the period of the loan has to be calculated.

Example

A bank agrees to give you a bond for a house which is on sale for R750 000. You can only afford to pay R8 500 per month. How long will it take to pay off the bond and what will the very last payment be if the bank agrees to a fixed rate of 12% p.a. compounded monthly?

Solution

We draw a timeline to illustrate the problem:



If we assume there will be exactly n payments of R8 500 then, using the formula for the present value of an annuity, $P_v = \frac{x[1-(1+i)^{-n}]}{i}$ we have

Present value $P_v = 750000$

Monthly payment $x = 8500$

Interest rate $\frac{i}{12} = 0,01$

Number of payments $= n$

$$\therefore\ 750000 = \frac{8500[1-(1+0,01)^{-n}}{0,01}$$

$$\therefore\ \frac{750000 \text{ x } 0,01}{8500} - 1 = -1,01^{-n}$$

$$1,01^{-n} = 0,117647...$$

$$-n\log 1,01 = \log 0,117647...$$

$$n = \frac{-\log 0,117647...}{\log 1,01} \approx 215,0749...$$

Thus, there are 215 payments of exactly R8 500 and one small payment of Ry a month later, the 216th payment. Since we have to pay interest on the final payment (just like all the others) we convert it to a present value of $y(1,01)^{-216}$ $(F_v = P_v(1+i)^n \therefore P_v = F_v(1+i)^{-n})$

So we now have:

$$750000 = \frac{8500[1-1,01^{-215}]}{0,01} + y(1,01)^{-216}$$

(Present value of 215 payments) (Present value of 216th payment)

$$\text{So}\quad y = [750000 - \frac{8500[1-1,01^{-215}]}{0,01}] \cdot 1,01^{216} = \text{R}639,37$$

There will be 216 payment in total (i.e. 18 years) and the final payment will be R639,37.

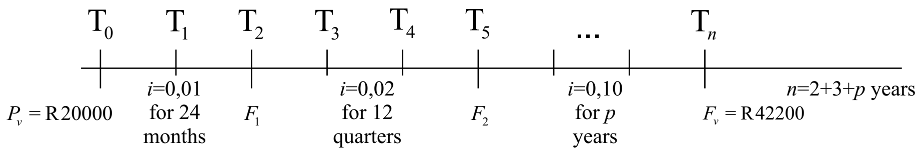
Fluctuating interest rates

In the real world, interest rates seldom remain fixed.

Example

R20 000 is invested in a bank for a number of years. The interest rate for the first two years is 12%p.a. compounded monthly. It then decreases to 8% p.a. compounded quarterly for the next three years, whereafter it increases again to 10% p.a. compounded yearly for the rest of the period. How many years in total was the R20 000 invested if it paid out R42 200 in the end?

Solution



$$F_1 = 20000(1+0,01)^{24},\ F_2 = F_1(1+0,02)^{12},\ F_v = V_2(1+0,10)p$$

$$\therefore F_v = 20000(1,01)^{24} \text{x} (1,02)^{12} \text{x} (1,10)^{p} = 42200$$

$$32206,61 \text{x} (1,10)^{p} = 42200$$

$$(1,10)^{p} = \frac{42200}{32206,61} = 1,310289...$$

$$p \log 1,10 = \log 1,310289...$$

$$p = \frac{\log 1,310289...}{\log 1,10} \approx 2,835...$$

$p = 2,84$ years (to 2 decimal places)

$\therefore$ Total number of years = 2+3+2,84=7,84 years.

CHECKLIST

Are you able to:

- work with logarithms
- calculate fixed payment annuities
- work with fluctuating interest rates
- solve exponential equations
- calculate compound interest

SELF-CHECK EXERCISE

1. Solve the following exponential equations. Give answers correct to 2 decimal places where applicable.

   a) $2^x = 12$ b) $3,5^{x+1} = 125$ c) $3^{-x+2} = 18$

2. A sum of money is invested for n years at 6,5% p.a. compounded monthly.

   a) What is the effective annual rate?
   b) How long will it take for the investment to double in value?

3. Ditebogo is amortising her car loan by paying R1 000 p.m. The loan was made to her at a rate of 18% p.a. compounded monthly. The bank informs her that the outstanding balance is R10 907,51. How many more payments must Ditebogo make to amortise the loan?

4. (Optional) A loan of R30 000 at 14% p.a. compounded monthly is paid back at R5 000 per month, plus a final payment to amortise the loan.

a) Calculate the number of payments of R5 000 that are required, excluding a smaller final payment.

b) Calculate the value of the payment that must be made a month after the final payment of R5 000.

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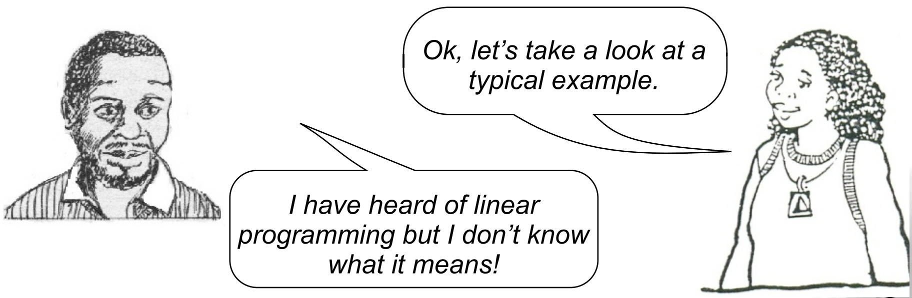
LESSON 6

Linear programming

About this lesson

Linear programming is a method of solving problems by transforming the problem into a set of inequalities and solving them graphically. Linear programming has many applications in everyday life. For example, it is very useful in agriculture, business, engineering and manufacturing. Even the way in which the robots (traffic lights) operate is based on linear programming.



Fire regulations require that a cinema should admit a maximum of 400 people. Each cheap seat occupies 1m^2 of space and each expensive seat occupies 2m^2 of space. The are 600m^2 of space available. Cheap seats cost R15 and expensive seats costs R20. The question is, 'what allocation of seats will earn the most money for the cinema owner?'

This is where linear programming helps us. We will not solve this problem just yet as you will need the algebra you studied in earlier units, especially linear equations, inequalities and simultaneous equations. You will also need some graph paper for this lesson.

In this lesson you will:

- represent systems of linear inequalities
- find the feasible region
- translate problems into a set of linear inequalities
- work with optimal problems and solutions

Representation of inequalities

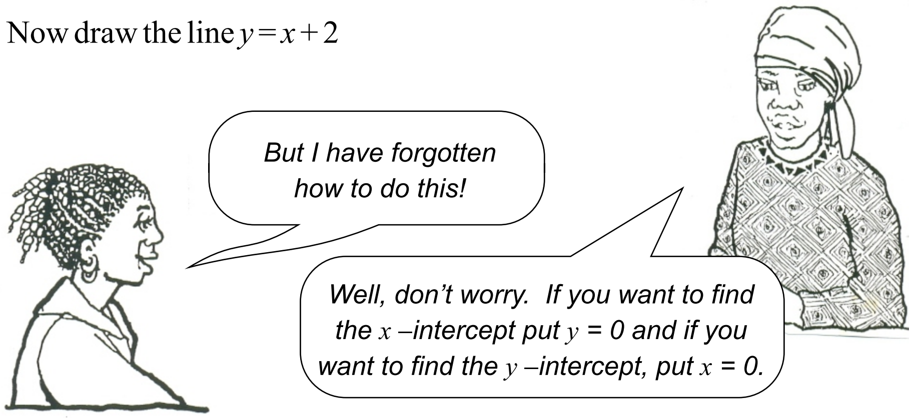
In Lesson 2 of Unit 2, you learned a lot about inequalities in general. In that lesson you learned that inequalities in two variables can be represented by regions on a plane. Let's look at an example to refresh your memory.

Example
Represent these two inequalities of two variables on the same system of axes: $y \geq -2x - 1$

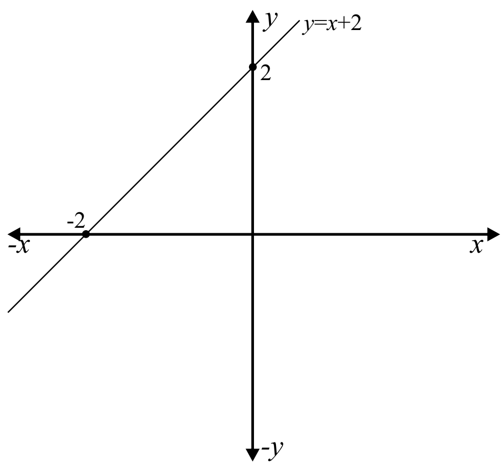
$y \leq x + 2$

Solution
To get the region of the inequality $y \leq x + 2$ on the plane, you first have to draw the boundary line. The boundary line of an inequality is the line with the inequality sign replaced by the equal sign. Therefore the boundary line of $y \leq x + 2$ is $y = x + 2$.

Now draw the line $y = x + 2$



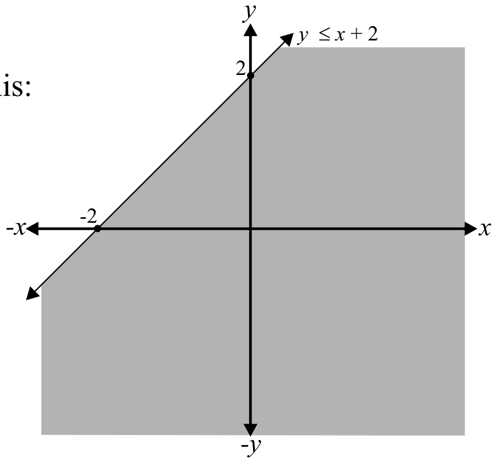
When $x = 0$, we get $y = 2$.
When $y = 0$, we get $0 = x + 2$ which gives $x = -2$.
This gives the line below:



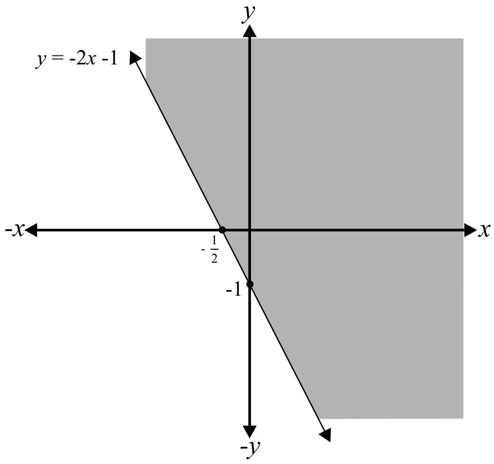
The second step is to find which side of the line satisfies the inequality. Because the line has divided the Cartesian plane into two half-planes, we have to check to see which side satisfies the inequality $y \leq x + 2$. It is usually helpful to start with the origin (0;0) as a set of ordered pairs and see if $y \leq x + 2$.

Well, $(0) \leq (0) + 2$ means that $0 \leq 2$ which we know is true therefore we shade all the points on the half-plane that are on the same side of the origin.

So our graph will now look like this:

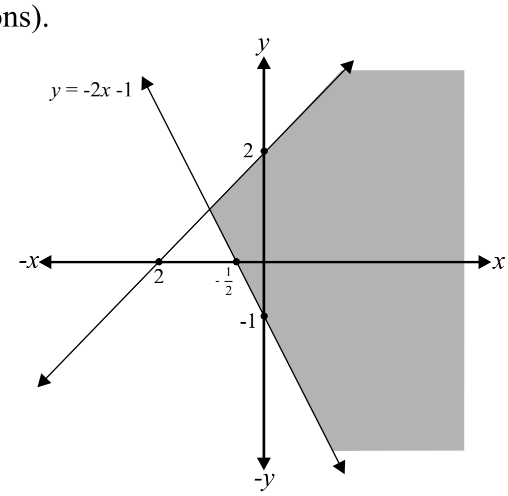


We do the same for the other inequality $y \geq -2x - 1$. The boundary line is $y = -2x - 1$. This gives the line below:



Put (0;0) in the inequality $y \geq -2x - 1$, and you will get $0 \geq -1$ which is true. Therefore we shade the side of the line with the point (0;0) as in the diagram above.

If we put the two regions on the same system of axes, we get the region in the diagram below. Note that the shaded region is the region where both inequalities are satisfied simultaneously (the intersection of the two regions).



ACTIVITY 1

Represent the following systems of linear inequalities on graph paper:

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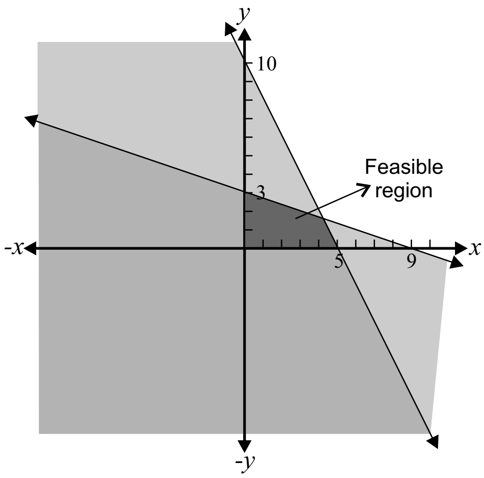
a) $2x + 3y < 1$
b) $x + 3y \leq 9,\ 2x + y \leq 10$

Feasible regions

What is the feasible region? We have already dealt with the feasible region except that we did not call it by this name. To start with, let's try to understand what we means by the words **feasible** and **region.** Given a system of inequalities, a point (x;y) is said to be **feasible** if it satisfies that system of inequalities. If we draw a line of inequality, it will divide the Cartesian plane into two areas.

We refer to either of these two areas as a **region.**

The feasible region of a system of inequalities is therefore the region which satisfies all the inequalities. In order to visualise this, let's take another look at the diagram in Activity 1b):



As you can see from the diagram above, the feasible region is the intersection of the two shaded regions. In this graph, it is the double-shaded region.

We call the inequalities used to draw the feasible region (or solve the given problem) constraints.

ACTIVITY 2

Suppose in addition to the inequalities in the first example (i.e. $y \leq x + 2$, $y \geq -2x - 1$) we also have the following inequalities: $x \geq 0$ and $y \geq 0$. Find the feasible region.

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Translating problems into a set of linear inequalities

You have learned to translate problems into a system of linear equations. Now we will extend this to cover inequalities. Let's start with one inequality involving two variables.

Example

Nocawa has R10,00 to spend on oranges. A big orange costs 75c while a small orange costs 50c. Find an inequality to show how many of each kind of orange she can buy.

Solution

Let x be the number of big oranges and y be the number of small oranges she can buy. Then the cost of the big oranges is $75x$ and the cost of the small oranges is $50y$. Therefore the total cost of the oranges is $75x + 50y$. But she has only 1000c (i.e. R10,00) for the oranges, so the total cost must be less than, or equal to, 1000. We can write this as an inequality:

$$75x + 50y \leq 1000$$

or $\quad 3x + 2y \leq 40 \quad$ (divide through by 25)

We also know that x cannot be a negative amount of big oranges so $x \geq 0$ and y cannot be a negative amount of small oranges so $y \geq 0$.

Therefore the full set of constraints for this problem are:

$$75x + 50y \leq 1000$$
$$x \geq 0$$
$$y \geq 0$$

ACTIVITY 3

Translate the following problems into inequalities involving two unknowns.

1. A patient can take his medicine either as tablets or as capsules. Each tablet contains 3 units of vitamin A, and each capsule contains 4 units. The daily intake of vitamin A must be at least 15 units.

2. A farmer wants to spend a maximum of R1 000,00 on goats and sheep. A goat costs R200,00 and a sheep R250,00. Find an equality to show how many goats and sheep he can buy.

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Let us now consider the situation where we have more than one inequality.

Example

A factory manager wants to buy two types of machine. Type 1 occupies $3m^2$ and costs R200, type 2 occupies $2m^2$ and costs R400. There is $40m^2$ of space available, and R4 000 to spend. If x is the number of type 1 machines and y the number of type 2 machines, show that the given information can be expressed by the inequalities:

$$3x + 2y \leq 40 \text{ and } x + 2y \leq 20$$

Solution

First of all, let's get the inequality for the restriction on space. Since there are x number of type 1 machines each occupying $3m^2$ of space the total space occupied by the type 2 machine is $2ym^2$. The space available for all the machines $40m^2$.

Therefore $3x + 2y \leq 40$.

Now let's get an inequality for the cost. What will the cost of the type 1 machines be?

It is R200x because each machine costs R200 and there are x of them.

What about the total cost for the type 2 machines? It is R400y.

Therefore the total cost for all the machines is R($200x + 400y$). This should be less than or equal to R4 000, because this is the money the manager can spend on the machines.

This gives the inequality: $200x + 400y \leq 4\,000$

Divide through by 200 to get: $x + 2y \leq 20$

So the two inequalities that summarise the information about the machines are: $3x + 2y \leq 40$ and
$x + 2y \leq 20$ and
$x \geq 0$ and $y \geq 0$

ACTIVITY 4

1. Go back to the example on fire regulations for a cinema in the Introduction of this lesson, and write down inequalities to represent the problem.

2. Food A contains 4 units of protein and 5 units of starch per kg, and food B contains 6 units of protein and 3 units of starch. The minimum daily intake of starch per person is 11 units. Let x be the number of kg of A to be eaten, and y the number of kg of B. Obtain two inequalities in x and y, one for the minimum amount of each food a person must eat to get the amount of protein she needs per day and the other for the amount of starch she needs per day.

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This leads to the next section. Most linear programming problems often ask for the minimum or maximum value of a quantity (i.e. the values of the variables or unknowns which give the maximum or minimum quantity). We are therefore going to see how we can use the idea of feasible regions to solve such problems.

Optimisation

Refer to the problem of the seats in the cinema. There you are asked to find the allocation of seats which gives maximum money. Sometimes you may rather be interested in finding the minimum quantity (say minimum amount of a given food you must eat to get a given amount of protein). Problems of this nature where you try to obtain a maximum or a minimum (optimum) quantity are called optimisation problems. The quantity whose optimum has been obtained is said to be optimised. The solutions obtained in optimisation problems (when a quantity has been optimised) are called optimal solutions.

An optimal solution is always obtained at a vertex (the point where any two of the lines of the set of inequalities meet) of the feasible region.

We can use the cinema seats problem to find a feasible region then we can explore the optimal solution problem.

Here is the question again: Fire regulations require that a cinema should admit a maximum of 400 people. Each cheap seat occupies $1m^2$ of space and each expensive seat occupies $2m^2$ of space. There is $600m^2$ of space available. Cheap seats cost R15 and expensive seats costs R20. The question is, 'what allocation of seats will earn the most money for the cinema owner?'

First we have to find the full set of inequalities for the problem. This should be:

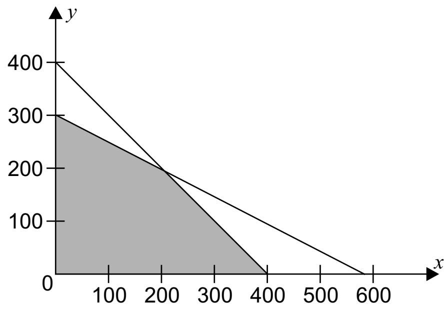
$$x = y \leq 400$$
$$x + 2y \leq 600$$
$$x \geq 0$$
$$y \geq 0$$

Next, we must find the objective function. The objective function is the function whose minimum or maximum value we wish to find.

What then is the objective function for the problem?

Remember that we were asked to find the number of cheap and expensive seats which will give maximum money. Therefore the objective function is the function that deals with money.

Let M be the money received by the cinema, then the objective function will be: M = 15x + 20y because there are x cheap seats at R15 each and y expensive seats at R20 each. Now to find the maximum value of the objective function, we need to draw the feasible region.

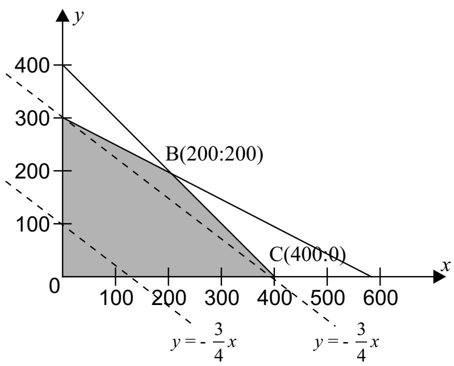


Although there are many points in the feasible region that will satisfy the objective function, we want to find the point that will give us the maximum amount of money. Let's look again at the objective function.

$$\begin{aligned} M &= 15x + 20y, \quad \text{which is a linear function so} \\ 20y &= -15x + M \\ y &= -\frac{15}{20}x + \frac{M}{5} \end{aligned}$$

The gradient of the objective function is $-\frac{3}{4}$ (divide $-\frac{15}{20}$ by 5) and we can draw a number of parallel lines which satisfy this constraint. Let's put this line on the graph to show you what happens.

The dotted line on this diagram is your objective function.



Substitute B(200:200) into

$$\begin{aligned} M &= 15x + 20y \\ M &= 15(200) + 20(200) \\ M &= 3000 + 4000 \\ M &= R7000 \end{aligned}$$

Substitute C(400;0) into

$$\begin{aligned} M &= 15x + 20y \\ M &= 15(400) + 200(0) \\ M &= 6000 + 0 \\ M &= R6000 \end{aligned}$$

Therefore the best allocation is point B; 200 cheap seats and 200 expensive seats which will give you the maximum total of R7 000.

ACTIVITY 5

1. Food A contains 4 units of protein and 5 units of starch per kg, and food B contains 6 units of protein and 3 units of starch. The minimum daily intake of protein per person must be 16 units, and the minimum daily intake of starch per person must be 11 units. Let x be the number of kg of food A to be eaten and y the number of kg of food B. What is the least weight of food (quantities of food A (x) and food B (y)) a person must take to ensure that he gets the required minimum daily intake of starch and proteins?

2. A farmer wants to spend a maximum of R1 000,00 on goats and sheep. A goat costs R200,00 and a sheep costs R250,00. The farmer knows that he can get a profit of R60,00 on each goat and R70,00 on each sheep. How many goats and sheep should the farmer buy in order to get the maximum profit?

ANSWERS ON PAGE 97

You have now completed this section so let's revise the procedure for solving a linear programming problem.

Summary

In this lesson you looked at a linear programming as a method of solving problems. You also learnt about inequalities in two variables that can be represented by regions on a plane and represent linear inequalities on graph paper.

You also dealt with feasible regions and you translated problems into a set of linear inequalities.

Finally you looked at problems where you tried to obtain a maximum or minimum quantity called optimal solutions.

CHECKLIST

Are you able to:

- draw the feasible region given a set of linear inequalities;
- find the vertices of a feasible region;
- translate a problem into a set of linear inequalities;
- find the objective function, optimal solutions and to use the objective function and the feasible region to obtain optimal solutions.

SELF-CHECK EXERCISE

1. Celiwe has a maximum of R1 500 to spend on pure woollen material and on synthetic material. She buys x metres of the woollen material at R1,50 per metre and y metres of the synthetic material at R2,00 per metre.

   a) Write down an inequality involving x and y for the above information.
   b) Write down any other constraints on x and y.
   c) Find the feasible region for the inequality.
   d) If the profit that Celiwe can make from 1 metre of woollen material is 40 cents and the profit on the synthetic material is 50 cents:

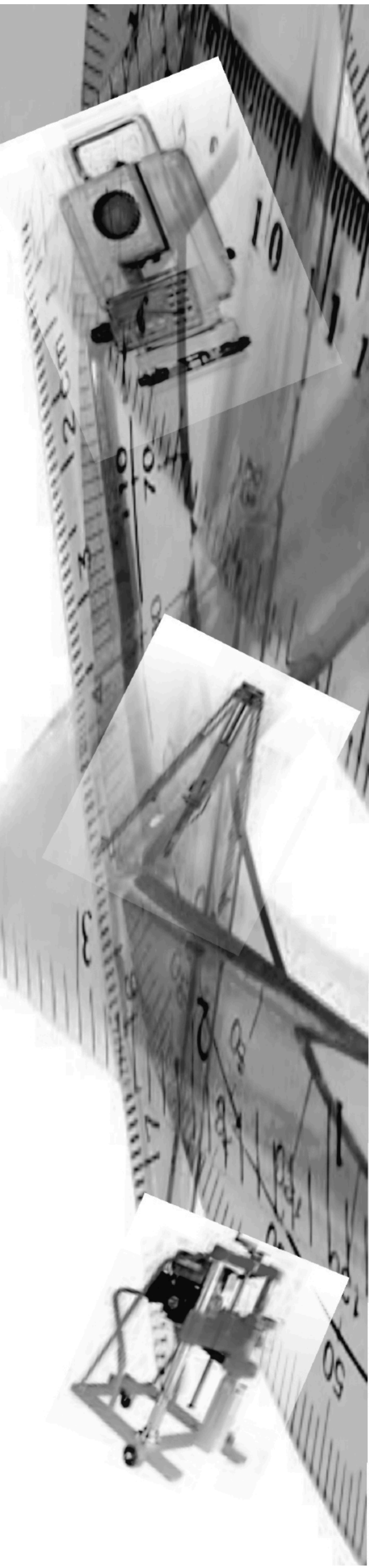
      i) write down the objective function.
      ii) what is the maximum possible profit that Celiwe can make?

2. A mail order firm must deliver 900 parcels using a lorry which takes 150 parcels at a time or a van which takes 80 parcels at a time. Each journey costs R50 for the lorry and R40 for the van. The total cost cannot exceed R450 and the lorry must not make more journeys than the van.

   a) Let x be the number of lorry journeys and y the number of van journeys. Find all the possible inequalities you can derive from the information given above.
   b) Draw the graph of the feasible region.
   c) What should the values of x and y be in order to make the total number of journeys as small as possible?

ANSWERS ON PAGE 111

Feedback to Activities



Lesson 1

Activity 1

1. All these are number patterns which are easy to follow. The next five terms in each of them are as follows:

   a) 6, 7, 8, 9, 10
   b) 11, 13, 15, 17, 19
   c) 12, 14, 16, 18, 20
   d) 21, 25, 29, 33, 37
   e) –10, –13, –16, –19, –22
   f) 64, 128, 256, 512, 1024
   g) 243, 729, 2 187, 6 561, 19 683

2. In the first five questions, we found the terms by adding a certain constant number to the previous number. This means we can find the common difference (d) between any two consecutive terms in the sequence. (Consecutive means following each other.)

   a) $d = 1$
   b) $d = 2$
   c) $d = 2$
   d) $d = 4$
   e) $d = -3$

   The next two patterns are different. There is no common, or constant, difference between consecutive terms. We can get the next term by multiplying the previous term by a certain number, a constant. This means that we can get this constant by dividing the second number by the first, the third by the second, and so on. This constant is called a common ratio, r.

   f) $r = 2$
   g) $r = 3$

   This will be clearer in the next section.

Activity 2

1. $T_k = 5 + (k - 1)6$

   a) The first term is found when $k = 1$. Therefore $T_1 = 5$. (We can also get the answer by remembering the formula for the general term, $T_k = a + (k - 1)d$. It is clear that $a = 5$. However the general term is seldom written in this form as it is usually simplified.)

   b) $T_2 = 5 + (2 - 1)6 = 11$
   $T_3 = 5 + (3 - 1)6 = 17$

c) The common difference, d = 6 because
$d = T_2 - T_1 = 11 - 5 = 6$ or
$d = T_3 - T_2 = 17 - 11 = 6$

2. $T_k = 4 + 2k$ is not in the conventional way of expressing a general term, but we can still find any term of this sequence.

a) $T_1 = 4 + 2 \times 1 = 6$.

b) First let us find T_2.
$T_2 = 4 + 2 \times 2 = 8$
$\therefore d = T_2 - T_1 = 8 - 6 = 2$
The common difference $d = 2$.

c) $T_k = 46$. We must find k.

$$\begin{aligned} 4 + 2k &= 46 \\ 2k &= 42 \\ k &= 21 \end{aligned}$$

46 is the 21st term of the sequence.

Activity 3

1. The general term of an arithmetic sequence is given by the formula $T_k = a + (k - 1)d$.

a) In this question we are given two terms. In order to find the general term we must find the first term, a, and the common difference, d. We can use the information given to make two simultaneous equations which we will solve to find the unknown values of a and d.

$$T_4 = a + 3d = -11 \qquad ①$$
$$T_{10} = a + 9d = -35 \qquad ②$$

$$②-①: \quad \begin{aligned} 6d &= -24 \\ d &= -4 \end{aligned}$$

Substitute for d in ① or ②: (it does not matter which we choose.)

In ①: $$\begin{aligned} a + 3(-4) &= -11 \\ a - 12 &= -11 \\ a &= 1 \end{aligned}$$

The general term is $T_k = 1 + (k - 1)(-4) = 5 - 4k$

The same method can be used to solve the others.

b) $$T_3 = a + 2d = 16 \qquad ①$$
$$T_6 = a + 5d = 37 \qquad ②$$

$$②-①: \quad \begin{aligned} 3d &= 21 \\ d &= 7 \end{aligned}$$

Substitute for d in ②: $a + 5x7 = 37$

$$a + 35 = 37$$

$$a = 2$$

The general term is $T_k = 2 + (k-1)7 = 7k-5$.

c) $T_4 = a + 3d = 5x + 8y$ ①

$T_{10} = a + 9d = 11x + 26y$ ②

②-①: $6d = 6x + 18y$

$$d = x + 3y$$

Substitute for d in ①:

$$a + 3(x + 3y) = 5x + 8y$$

$$a = 5x - 3x + 8y - 9y$$

$$a = 2x - y$$

The general term is $T_k = 2x-y + (k-1)(x + 3y)$
or $T_k = x-4y + (x + 3y)\,k$

2. If you look at the three given terms in this question, you will find that they follow each other, with nothing between them. Remember, when terms follow each other, like 1; 2; 3; and so on, we call them **consecutive terms**.

a) In this question we are given three consecutive terms of a sequence. We can easily find x, because we construct two equations that are equal:

$$T_2 - T_1 = d$$

$$T_3 - T_2 = d$$

Therefore, $T_3 - T_2 = T_2 - T_1$

$$(4x + 7) - (3x + 4) = (3x + 4) - (x + 3)$$

$$4x + 7 - 3x - 4 = 3x + 4 - x - 3$$

$$x + 3 = 2x + 1$$

$$x = 2$$

Therefore, $a = T_1 = 2 + 3 = 5$

$$T_2 = 3\text{x}2 + 4 = 10$$

$$d = T_2 - T_1 = 5$$

The sequence is 5; 10; 15; 20; 25; ...

b) The general term is $Tk = 5 + (k - 1)5 = 5k$

c) $T_{50} = 250$

You will note that in most cases we simplify the general term. It is necessary to have the general term in a simpler form in order to get any term easily. The last problem has shown this advantage.

Activity 4

1. First write down the general term of any arithmetic sequence.

$$T_k = a + (k-1)d$$

$$T_1 + T_2 + T_3 = a + (a+d) + (a+2d)$$

$$= 3a + 3d$$

and

$$T_1 \times T_2 = a(a+d)$$

We have been told that $T_1 + T_2 + T_3 = 3a + 3d = 12$ ①

and $T_1 \times T_2 = a(a+d) = 8$ ②

From equation ①: $3a = 12 - 3d$

$a = 4 - d$ ③

Substitute in ②: $(4-d)(4-d+d) = 8$

$$16 - 4d = 8$$

$$-4d = -8$$

$$d = 2$$

Therefore $a = 2$ (from ③)

The sequence is 2; 4; 6; 8; ...

2. Let the first mass be equal to a units. Let the common difference between the masses be d, then

$$T_1 = a$$

$$T_{11} = a + 10d$$

but $T_{11} = 9 \times T_1$

$$\therefore a + 10d = 9a$$

$$8a - 10d = 0$$

$4a - 5d = 0$ ①

also, $T_7 = a + 6d$

$$T_3 = a + 2d$$

and $T_7 = 2 \times T_3 + 1$

$$\therefore a + 6d = 2(a+2d) + 1$$

$a - 2d = -1$ ②

① $-4 \times$ ② : $3d = 4$

$$d = \frac{4}{3}$$

From ②: $a = 2d - 1$

$$= \frac{8}{3} - 1$$

$$= \frac{5}{3}$$

The sequence of masses is:

$$\frac{5}{3};3;\frac{13}{3};\frac{17}{3};7;\frac{25}{3};\frac{29}{3};11;\frac{37}{3};\frac{41}{3};15;\frac{49}{3};\frac{53}{3};19;\frac{61}{3};\frac{65}{3};23;\frac{73}{3};\frac{77}{3};27.$$

Activity 5

1.
$$\begin{aligned} T_5 &= ar^4 \\ a \times 5^4 &= 1875 \\ 625a &= 1875 \\ a &= 3 \\ T_1 &= 3 \\ T_2 &= 3 \times 5^1 = 15 \\ T_3 &= 3 \times 5^2 = 75 \end{aligned}$$

2.
$$\begin{aligned} T_1 &= a = 4 \\ T_4 &= ar^3 = 0,846 \\ 4r^3 &= 0,864 \\ r^3 &= 0,216 \\ r &= 0,6 \qquad \text{(use a calculator)} \end{aligned}$$

Therefore, $T_6 = ar^5 = 4 \times (0,6)^5 = 0,311\ 04$

3.
$$\begin{aligned} T_4 &= ar^3 = 24(x-y)^4 \\ T_7 &= ar^6 = 192(x-y)^7 \\ \frac{T_7}{T_4} &= \frac{ar^6}{ar^3} = \frac{192(x-y)^7}{24(x-y)^4} \\ \therefore r^3 &= 8(x-y)^3 \end{aligned}$$

Therefore, $r = 2(x-y)$

So, $T_4 = a\left[2(x-y)\right]^3 = 24(x-y)^4$

$$a = \frac{24(x-y)^4}{8(x-y)^3} = 3(x-y)$$

4.

$$r = \frac{T_2}{T_1} = \frac{T_3}{T_2}$$

$$\therefore \frac{x-2}{x+2} = \frac{x}{x-2}$$

$$(x-2)^2 = x(x+2)$$

$$x^2 - 4x + 4 = x^2 + 2x$$

$$6x = 4$$

$$x = \frac{2}{3}$$

$$\therefore T_1 = \frac{8}{3}$$

$$T_2 = -\frac{4}{3}$$

$$T_3 = \frac{2}{3}$$

$$r = \frac{T_2}{T_1}$$

$$\therefore r = -\frac{4}{3} \cdot \frac{3}{8}$$

$$\therefore r = -\frac{1}{2}$$

Activity 6

a) $\frac{T_2}{T_1} = \frac{-6}{3} = -2$

$\frac{T_3}{T_2} = \frac{12}{-6} = -2$

This is a geometric sequence.

b) $T_2 - T_1 = -9 - (-15) = -9 + 15 = 6$

$T_3 - T_2 = -3 - (-9) = -3 + 9 = 6$

This is an arithmetic sequence.

c) $T_2 - T_1 = bx^2 - x = x(bx - 1)$

$T_3 - T_2 = b^2x^3 - bx^2 = x^2(bx - 1)$

This is clearly not an arithmetic sequence.

$\frac{T_2}{T_1} = \frac{bx^2}{x} = bx$

$\frac{T_3}{T_2} = \frac{b^2x^3}{bx^2} = bx$

This is a geometric sequence.

Activity 7

a) The sequence appears to be the squares of the natural numbers ($1^2 = 1$; $2^2 = 4$; $3^2 = 9$; $4^2 = 16$). So, the next three terms will be $5^2 = 25$; $6^2 = 36$; $7^2 = 49$. A formula for the general term could be $T_k = k^2$.

b) This sequence appears to be the reciprocals of the natural numbers ($\frac{1}{1}$; $\frac{1}{2}$; $\frac{1}{3}$; $\frac{1}{4}$; etc),

in which case the next three terms will be $\frac{1}{5}$; $\frac{1}{6}$; $\frac{1}{7}$.

The general term could be written $T_k = \frac{1}{k}$.

c) It seems obvious that the next three terms of this sequence are: 11111; 111111; 1111111. A formula for the general term, however, is not very obvious. Here are two possible formulae that you may wish to play with:

(i) $T_k = 10^{k-1} + 10^{k-2} + 10^3 + ... + 10^0$

(ii) $T_1 = 1$ and $T_{k+1} 10^k + T_k$ for $k = 1; 2; 3; ...$

d) There does not even appear to be an obvious pattern developing in this sequence. In fact each term is the number π (pi) expressed to one more decimal than its predecessor. By pressing the π button on your calculator you will find that the next three terms are 3,14159; 3,141592; 3,1415926. Don't even bother trying to find a simple formula for the general term - nobody has ever succeeded.

Convince yourself that the above sequences are neither arithmetic nor geometric.

Activity 8

a) $T_1 = \frac{1}{1+1} = \frac{1}{2}$; $T_2 = \frac{2}{2+1} = \frac{2}{3}$; $T_3 = \frac{3}{3+1} = \frac{3}{4}$; $T_{20} = \frac{20}{20+1} = \frac{20}{21}$

b) $T_1 = (-1)^1 \text{ x } 1 = -1$; $T_2 = (-1)^2 \text{ x } 2 = 2$; $T_3 = (-1)^3 \text{ x } 3 = -3$;

$T_{20} = (-1)^{20} \text{ x } 20 = 20$

This is an 'alternating sequence' because the signs of the terms alternate between – and +.

c) $T_1 = \frac{(-1)^2}{1^2} = +1$; $T_2 = \frac{(-1)^3}{2^2} = \frac{-1}{4}$; $T_3 = \frac{(-1)^4}{3^2} = \frac{1}{9}$;

$T_{20} = \frac{(-1)^{21}}{20^2} = \frac{-1}{400}$

Another alternating sequence.

Lesson 2

Activity 1

It is usually an advantage to write down at least the first three terms of a series. This helps you to recognise the first term and the common difference easily.

1. $1+2+3+4+...$ This is an arithmetic series with:

   $a=1,\ d=1,\ n=100$

   $$S_n=\frac{n}{2}\left[2a+(n-1)d\right] \qquad \text{or} \qquad S_n=\frac{n}{2}\{a+l\}$$

   $$\therefore\ S_{100}=\frac{100}{2}\left[2(1)+(100-1)1\right] \qquad \therefore S_{100}=\frac{100}{2}\{1+100\}$$

   $$=50(2+99) \qquad =50.101$$

   $$=5\ 050 \qquad =5\ 050$$

2. $2+4+6+...$ This is an arithmetic series with:

   $a\ =\ 2,\ d\ =\ 2,\ n\ =100$

   $$S_n=\frac{n}{2}\left[2a+(n-1)d\right] \qquad \text{or} \qquad S_n=\frac{n}{2}\{a+l\}$$

   $$\therefore\ S_{100}=\frac{100}{2}\left[2(2)+(100-1)2\right] \qquad \therefore S_{100}=\frac{100}{2}\{2+200\}$$

   $$=50\left[4+99(2)\right] \qquad =50.202$$

   $$=50(4+198) \qquad =10\ 100$$

   $$=50\times 202$$

   $$=10\ 100$$

3. $1+3+5+...$ This is an arithmetic series with:

   $a=1,\ d=2,\ n=100$

   $$S_n=\frac{n}{2}\left[2a+(n-1)d\right]$$

   $$S_{100}=\frac{100}{2}\left[2(1)+(100-1)2\right]$$

   $$=50(2+198)$$

   $$=10\ 000$$

4. This is an arithmetic series with:

   $a=1,\ d=1\frac{1}{3},\ n=20$

$$S_n = \frac{n}{2}\left[2a+(n-1)d\right]$$

$$\therefore S_{20} = \frac{20}{2}\left[2(1)+(20-1)\frac{4}{3}\right]$$

$$=10(2+\frac{76}{3})$$

$$=10 \times \frac{82}{3}$$

$$=273\frac{1}{3}$$

Activity 2

1. There are many questions like this one. It is usually not clear to people that this is just another series problem. It becomes clearer if one is able to list a few terms of the series to be able to recognise the pattern.

$$S_{24} = 2+4+8+...$$

This is a geometric series with a=2,

$$r = 2$$

$$n = 24$$

$$S_n = \frac{1(r^n - 1)}{r-1} \qquad (r>1)$$

$$\therefore S_{24} = \frac{2(2^{24}-1)}{2-1}$$

$$= 2(16\ 777\ 216 - 1)$$

$$= 33\ 554\ 430$$

The final instalment is $T_{24} = 2.2^{23} = 2^{24} = \text{R}16\ 777\ 216!$

2. $a = 6,\ r = \frac{1}{2},\ n = 15$

$$S_n = \frac{1(1-r^n)}{1-r} \qquad (r<1)$$

$$\therefore S_{15} = \frac{6\left(1-\left(\frac{1}{2}\right)^{15}\right)}{1-\frac{1}{2}}$$

$$= 6 \times \frac{32\ 767}{32\ 768} \times 2$$

$$= 6 \times \frac{32\ 767}{16\ 384}$$

$$= \frac{98\ 301}{8\ 192}$$

There is not much that one needs to remember for these series. It is only the formulas that you have to know.

3. $T_2 = ar = 6$ ①

$T_5 = ar^4 = 162$ ②

$\therefore \frac{T_5}{T_2} = \frac{ar^4}{ar} = r^3 = \frac{162}{6} = 27$ ② ÷ ①

$\therefore r = 3$

From ①, $a = 2$

$S_n = \frac{a(r^n - 1)}{r - 1}$

$\therefore S_{20} = \frac{2(3^{20} - 1)}{3 - 1} = 3^{20} - 1$

There is no need to expand 3^{20} .

Activity 3

1. $a = 1,\ r = -\frac{1}{2} \quad \therefore \ S_\infty = \frac{a}{1-r} = \frac{1}{1-(-\frac{1}{2})} = \frac{2}{3}$

2. 0,333...=0,3+0,03+0,003+0,0003+...
This is an infinite geometric series with

$a = 0,3$ and $r = \frac{0,03}{0,3} = \frac{1}{10} \quad \therefore \ S_\infty = \frac{0,3}{1-\frac{1}{10}} = \frac{3}{10} \times \frac{10}{9} = \frac{3}{9} = \frac{1}{3}$

3. A geometric series converges (i.e. has a sum to infinity) only if $-1 < r < 1$.
For this series $a = x$ and $r = \frac{2x^2}{x} = 2x$

So it converges when $-1 < 2x < 1$ i.e. $-\frac{1}{2} < x < \frac{1}{2}$

Its sum is $S_\infty = \frac{x}{1-2x}$ (but only when $-\frac{1}{2} < x < \frac{1}{2}$).

Activity 4

1. a) $\sum_{n=1}^{5} n^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2$

b) $\sum_{k=1}^{10} (2k - 1) = (2.1 - 1) + (2.2 - 1) + (2.3 - 1) + (2.4 - 1)$

$+ (2.5 - 1) + (2.6 - 1) + (2.7 - 1) + (2.8 - 1)$

$+ (2.9 - 1) + (2.10 - 1)$

$= 1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19$

c) $\sum_{r=0}^{5} \frac{1}{r+1} = \frac{1}{0+1} + \frac{1}{1+1} + \frac{1}{2+1} + \frac{1}{3+1} + \frac{1}{4+1} + \frac{1}{5+1}$

$= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}$

d) $\sum_{n=1}^{12} 2 = 2+2+2+2+2+2+2+2+2+2+2+2 (= 24)$

(i.e. simply the number 2 12 times).

e) $\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots \quad (= \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1)$

2. First find out whether the series is an arithmetic or geometric series and then find the general term.

a) arithmetic series, $a = 1, d = 4, n = 20$

$T_n = 1 + (n-1)4$

$= 4n - 3$

$\therefore 1 + 5 + 9 + \ldots$ to 20th term

$= \sum_{n=1}^{20} (4n-3)$

$\sum_{n=1}^{20} (4n-3) = \frac{20}{2}[2(1) + (20-1)4]$

$= 10(2 + 76)$

$= 780$

b) arithmetic series, $a = 4, d = 3$

To find n: let 31 be the last term, where

$T_n = a + (n-1)d = 4 + (n-1)3$

$= 3n + 1$

$\therefore 31 = 3n + 1$

$n = 10$

$4 + 7 + 10 + \ldots + 31 = \sum_{n=1}^{10} (3n+1)$

$S_{10} = \frac{10}{2}(4 + 31) = 175$

c) geometric series, $a = 10,\ r = \frac{1}{2},\ n = 10$

$\therefore T_n = 10.\left(\frac{1}{2}\right)^{n-1}$

$\therefore$ Series $= \sum_{n=1}^{10} 10\left(\frac{1}{2}\right)^{n-1}$

$$\sum_{n=1}^{10} 10\left(\frac{1}{2}\right)^{n-1} = S_{10} = \frac{10\left[1-\left(\frac{1}{2}\right)^{10}\right]}{1-\frac{1}{2}}$$

$$= 10\left(1-\frac{1}{1024}\right) \times 2$$

$$= 20 \times \frac{1023}{1024} \approx 19,98$$

d) geometric series, $a = 2,\ r = \frac{5}{2},\ n = 10$

$$T_n = 2\left(\frac{5}{2}\right)^{n-1}$$

$$\text{Series } = \sum_{n=1}^{10} 2\left(\frac{5}{2}\right)^{n-1}$$

$$\sum_{n=1}^{10} 2\left(\frac{5}{2}\right)^{n-1} = S_{10} = \frac{2\left[\left(\frac{5}{2}\right)^{10} - 1\right]}{\frac{5}{2}-1}$$

$$= \frac{5^{10} - 2^{10}}{2^9 \times 3} \times \frac{2}{3}$$

$$= \frac{5^{10} - 2^{10}}{2^8 \times 3}$$

$$= \frac{9\ 764\ 601}{768}$$

$$= 12\ 714,32 \text{ to 2 dec.}$$

e) This is an infinite geometric series with $a = 1$ and $r = \frac{1}{3}$.

$$T_n = 1.\left(\frac{1}{3}\right)^{n-1} = 3^{1-n}$$

$$\text{So series} = \sum_{n=1}^{\infty} 3^{1-n} \quad \left(\text{or } \sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^{n-1}\right)$$

$$S_\infty = \frac{1}{1-\frac{1}{3}} = \frac{3}{2}$$

Lesson 3

Activity 1

a) $P = 500,\ i = 12\% \text{ or } \frac{12}{100},\ n = 5$

$$A = P(1 + n.i)$$

$$= 500(1 + 5 \cdot 0,12)$$

$$= R800,00 \qquad \text{(R300 interest)}$$

b) $A = P(1+i)^n$

$= 500(1+\frac{12}{100})^5$

$= 500(1,762)$

$= R881,17$ (to 2 dec.) (R381,17 interest)

Activity 2

Each year the population grows by 1,8%, so this is just like compound interest.

$$P = 25 \text{ million}, \ i = 1,8\% \text{ or } \frac{18}{1000}, \ n = 32$$

$$A = P(1+i)^n$$

$$= 25(1+\frac{18}{1000})^{32} \quad (1,8\% = \frac{1,8}{100} = 0,018)$$

$$= 44,25 \text{ million} \quad \text{(to 2 dec.)}$$

Activity 3

$$P = R1\ 000, \ i = 5\% = 0,05, \ n = 10, \ A = P(1+\frac{i}{t})^{nt}$$

a) $t = 1 \therefore A = 1\ 000(1+\frac{0,05}{1})^{10.1} = R1\ 628,89$ ($R628,89$ interest)

b) $t = 2 \therefore A = 1\ 000(1+\frac{0,05}{2})^{10.2} = R1\ 638,62$ ($R638,62$ interest)

c) $t = 12 \therefore A = 1\ 000(1+\frac{0,05}{12})^{10.12} = R1\ 647,01$ ($R647,01$ interest)

d) $t = 365 \therefore A = 1\ 000(1+\frac{0,05}{365})^{10.365} = R1\ 648,66$ ($R648,66$ interest)

Activity 4

a) $P = R45\ 000, \ i = 4\% = 0,04, \ n = 4$

$A = P(1 - in)$

$= 45\ 000(1 - 0,04.4)$

$= R37\ 800$ This is the usual method used to calculate depreciation of an asset.

b) $A = R45\ 000, \ i = 4\% = 0,04, \ n = 4, \ t = 2$

$A = P(1-\frac{i}{t})^{nt}$ (Formula for depreciation adjusted for the time period)

$= 45\ 000\left(1 - \frac{0,04}{2}\right)^8$

$= 45\ 000(0,98)^8$

$\approx R38\ 284$ (To the nearest rand)

The car will be worth approximately R38 284 in four years time.

Activity 5

1. a) 5,25% p.a. compounded daily

   b) $A = 5\ 000(1+\frac{0,0525}{365})^{365} = R5\ 269,49$

   c) Suppose the effective rate is j% p.a. then the final amount must be the same, so

$$5\ 000(1+j.1)^1 = 5\ 000(1+\frac{0,052\ 5}{365})^{365}$$

$$1+j = (1+\frac{0,052\ 5}{365})^{365} \quad \text{(The principal always cancels)}$$

$$j = (1+\frac{0,052\ 5}{365})^{365} - 1$$

$$= 0,053\ 9 \quad \text{(to 4 decimals)}$$

$$= 5,39\%$$

2. In each case we must have $P(1+j) = P(1+\frac{i}{t})^t$ since $n = 1$

   so $j = (1+\frac{i}{t})^t - 1$ where i = 0,09 and j is the effective rate.

   a) $j = (1+\frac{0,09}{2})^2 - 1 = 0,0920 \therefore j = 9,20\%$ p.a.

   b) $j = (1+\frac{0,09}{4})^4 - 1 = 0,0931 \therefore j = 9,31\%$ p.a.

   c) $j = (1+\frac{0,09}{12})^{12} - 1 = 0,0938 \therefore j = 9,38\%$ p.a.

Activity 6

a) We want the future value of the annuity at the end of 20 years i.e. one month after the 240th payment.

Immediately after the 240th payment (20 x 12)

$$F_V = \frac{x[(1+i)^n - 1]}{i}$$

where $x = 250,\ i = \frac{6\%}{12} = \frac{0,06}{12}$ and $n = 240$

$$F_V = \frac{250[(1+\frac{0,06}{12})^{240} - 1]}{\frac{0,06}{12}} = R115\ 510,22$$

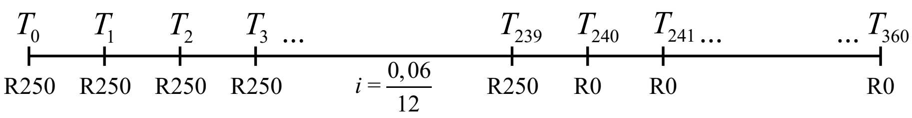
This money remains in the fund and accrues interest for another month, so at the end of 20 years she receives:

$$A = R115\ 510,22(1+\frac{0,06}{12})$$

$$= R116\ 087,77$$

b) If she leaves the money in the fund for another 10 years (at the same interest rate – which never happens in real life) she will eventually get:

$$A = R116\ 087,77(1+\frac{0,06}{12})^{120}$$
$$= R211\ 209,71$$



Activity 7

1. $P_V = \frac{x[1-(1+i)^{-n}]}{i}$ where $x = 3000,\ i = \frac{0,125}{12},\ n = 240$

$$= \frac{3\ 000[1-(1+\frac{0,125}{12})^{-240}]}{\frac{0,125}{12}} = R264\ 051,84 \text{ bond}$$

2. Here we must calculate x given $P_V = 15\ 000,\ i = \frac{0,15}{12}$ and $n = 10$:

$$15\ 000 = \frac{x[1-\left(1+\frac{0,15}{12}\right)^{-10}]}{\frac{0,15}{12}}$$

$$x = \frac{15\ 000.\left(\frac{0,15}{12}\right)}{1-\left(1+\frac{0,15}{12}\right)^{-10}} = R1\ 605,05 \text{ per month}$$

3. We must calculate P_v given

$$x = 1\ 000,\ i = \frac{0,045}{12},\ n = 12$$

$$P_v = \frac{1000[1-\left(1+\frac{0,045}{12}\right)^{-12}]}{\frac{0,045}{12}} = R11\ 712,55$$

4. a) $1 + j = (1+\frac{0,10}{365})^{365} \therefore j = 0,1\ 052 = 10,52\%\text{p.a.}$

b) The future value of the annuity is R4 000 000. We must calculate P_v given.
$x = 160\ 000,\ i = 10,52\%$ (**not** 10% because the compounding period and pay out period must be the same: 1 year)

$$\text{so } i = \frac{0{,}105\ 2}{1},\ n = 25$$

$$P_v = \frac{160\ 000[1-(1{,}105\ 2)^{-25}]}{0{,}105\ 2} = R1\ 396\ 150{,}55 \text{ (to invest)}$$

So, on the day of the draw the Lotto company needs
R1 000 000 + R1 396 150,55 = R2 396 150,55

5. Method 1:

We know the future value is $F_v = \dfrac{x[(1+i)^n - 1]}{i}$

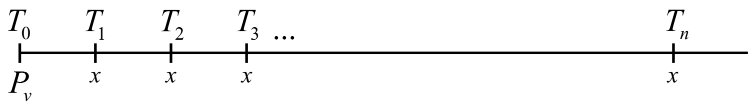
But the future value is the present value invested at *i* compound interest for *n* years so.

$$\therefore P_V (1+i)^n = \frac{x[(1+i)^n - 1]}{i}$$

$$P_V = \frac{x[(1+i)^n - 1]}{i} \cdot (1+j)^{-n}$$

$$= \frac{x[1-(1+i)^{-n}]}{i}$$

Method 2:



The present value of the 1st payment is $x(1+i)^{-1}$

The present value of the 2nd payment is $x(1+i)^{-2}$

The present value of the nth payment is $x(1+i)^{-n}$

$$\therefore P_v = x(1+i)^{-1} + x(1+i)^{-2} + x(1+i)^{-n} + \ldots + x(1+i)^{-n}$$

This is a geometric series with *n* terms and

$$a = x(1+i)^{-1},\ r = (1+i)^{-1}$$

$$\therefore P_v = S_n = \frac{a(1-r^n)}{1-r}$$

$$= \frac{x(1+i)^{-1}[1-(1+i)^{-n}]}{1-(1+i)^{-1}}$$

$$= \frac{x[1-(1+i)^{-n}]}{(1+i)\cdot[1-\frac{1}{1+i}]}$$

$$= \frac{x[1-(1+i)^{-n}]}{\frac{(1+i)\cdot i}{1+i}}$$

$$= \frac{x[1-(1+i)^{-n}]}{i}$$

6. a) We must calculate x given $P_v = 10\ 000,\ i = \frac{0,15}{12},\ n = 24$

$$10\ 000 = \frac{x[1-(1+\frac{0,15}{12})^{-24}}{\frac{0,15}{12}} = 20,624\ 2...x$$

$$x = R484,87$$

b) The number of payments remaining = 24 −10 =14

$$\therefore \text{ outstanding balance } = P_v = \frac{484,74[1-(1+\frac{0,15}{12})^{-14}}{\frac{0,15}{12}}$$

$$= R6\ 190,40$$

Activity 8

1. a) We must use the **buying** column as the bank is buying the currency from you.

b) $\text{R}1,00 = ¥11,113\,2 \ \therefore \ ¥ = \frac{1}{11,113\ 2} = \text{R}0,089\,98$ (i.e. 9c)

c) $€1\,520 = 1\,520 \times 9,767\,7 = \text{R}14\,846,90$

$£725 = 725 \times 11,167\,3 = \text{R}8\,096,29$

$¥16\,000 = \frac{16\ 000}{11,113\ 2} = \text{R}1\,439,73$

$\therefore$ total = R24 382,92

2. a) The bank is **selling** the currency to you.

b) \$1 000 will cost R7 310 (1 000 x 7,310)
so you will buy R7 310 + 3 x R2 000 = R13 310
worth of foreign currency.

The bank charges 1% commission $= \frac{1}{100}$ x13 310 = R133,10

You will end up paying R13 310,00 + R133,10 = R13 443,10

c) Pula: 2 000 x 0,869 5 = 1 739

Kwacha: 2 000 x 22,1500 = 44 300

Kenyan shillings: 2 000 x 12,5000 = 25 000

3. a) The bank buys 1 000 Australian Dollars for

$$\frac{1000}{0,1368} = \text{R}7\ 309,94$$

then sells 7 309,94 x 6,248 2 = 45 675,43 Rupees to Sipho.

b) The bank sells $\frac{7\ 309,94}{10,290\ 9}$ = € 710,33 to Sipho.

c) The bank buys the Euros back at R9,7677 to €1 so Sipho gets 710,33 x 9,7677 = R6 938,29
He has lost R7 309,94 – R6 938,29 = R371,65

Lesson 4

Activity 1

1. a) $81^{\frac{1}{2}} = \sqrt{81} = 9$ b) $64^{\frac{1}{3}} = \sqrt[3]{64} = 4$

c) $81^{\frac{1}{4}} = \sqrt[4]{81} = 3$ d) $16^{\frac{1}{4}} = \sqrt[4]{16} = 2$

You could also solve 1. a) using this approach. Therefore you have two ways of solving these problems.

2. a) $8^{-\frac{1}{3}} = \frac{1}{8^{\frac{1}{3}}} = \frac{1}{\sqrt[3]{8}} = \frac{1}{2}$ b) $9^{-\frac{1}{2}} = \frac{1}{9^{\frac{1}{2}}} = \frac{1}{\sqrt{9}} = \frac{1}{3}$

c) $32^{-\frac{1}{5}} = \frac{1}{32^{\frac{1}{5}}} = \frac{1}{\sqrt[5]{32}} = \frac{1}{2}$ d) $64^{-\frac{1}{3}} = \frac{1}{64^{\frac{1}{3}}} = \frac{1}{\sqrt[3]{64}} = \frac{1}{4}$

Activity 2

a) $2^{x-1} = 1$

You know that $1 = 2^0$

So in our example we say $1 = 2^0$

Therefore $2^{x-1} = 2^0$

gives $x - 1 = 0$ and $x = 1$

b) $3^x = \frac{1}{27}$

$3^x = \frac{1}{3^3}$

$3^x = 3^{-3}$ and $x = -3$

c) $2^{x+1} = (0,5)^{x-2}$

$2^{x+1} = \left(\frac{1}{2}^{x-2}\right)$

$2^{x+1} = (2^{-1})^{x-2}$

$2^{x+1} = 2^{-x+2}$

$x + 1 = -x + 2$

$x + x = 2 - 1$

$2x = 1$

$x = \frac{1}{2}$

d) $27^{x-2} = 81^{2x+1}$

$(3^3)^{x-2} = (3^4)^{2x+1}$

$3^{3x-6} = 3^{8x+4}$

$3x - 6 = 8x + 4$

$5x = -10$

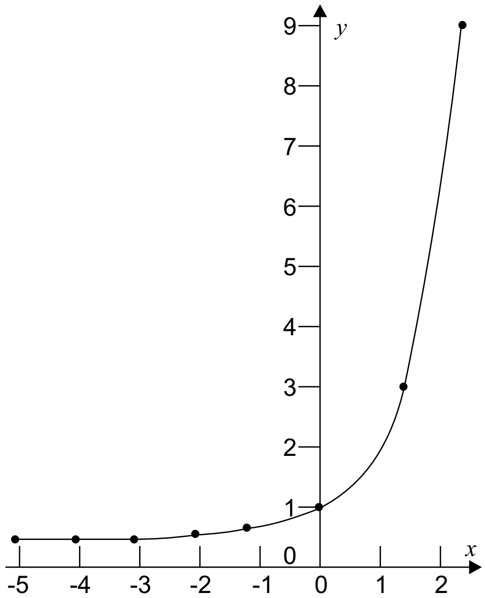
$x = -2$

Activity 3

1. a)

x	-5	-4	-3	-2	-1	0	1	2
y	0,0041	0,0123	0,037	0,1111	0,3333	1	2	9

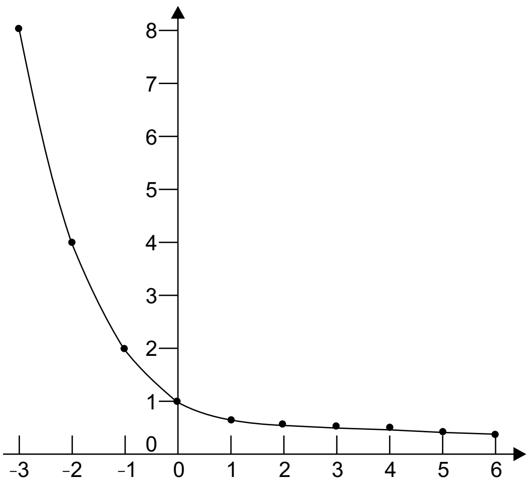
b) Graph of $y = 3^x$.



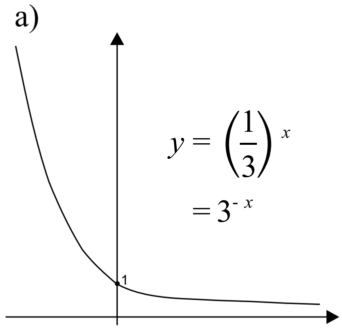
2. a)

x	–3	–2	–1	0	1	2	3	4	5
y	8	4	2	1	0,5	0,25	0,125	0,063	0,031

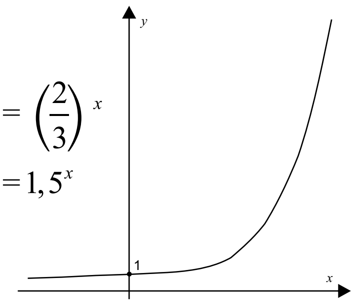
b) Graph of $y = \left(\frac{1}{2}\right)^x$.



3. a)

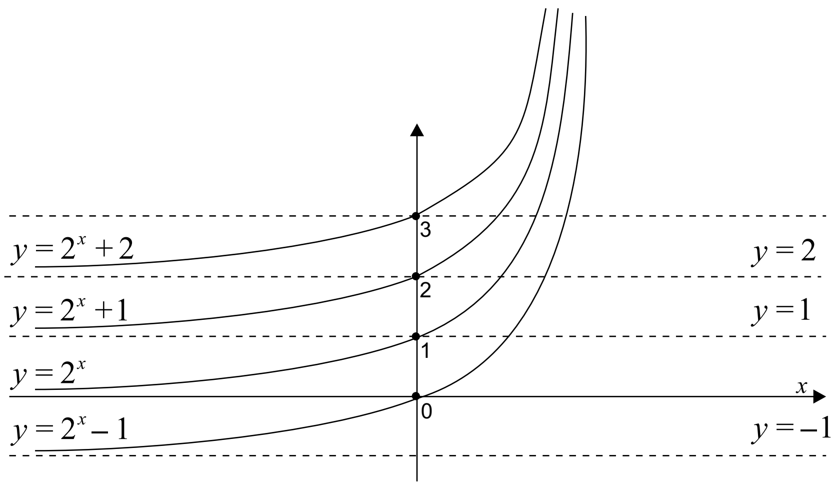


b)

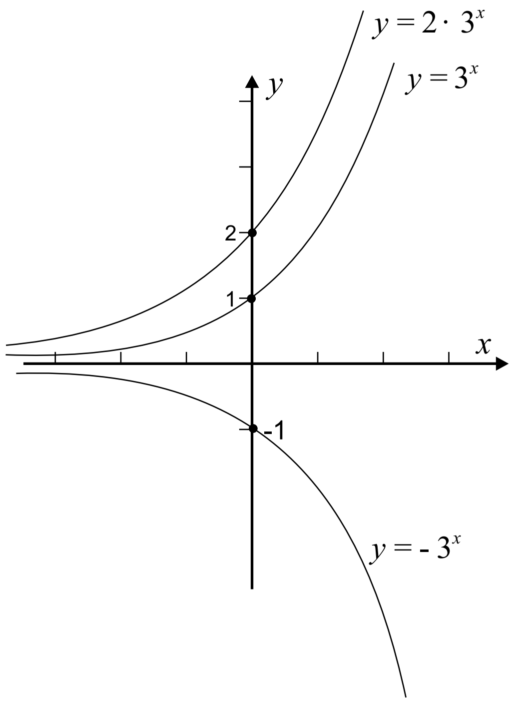


Activity 4

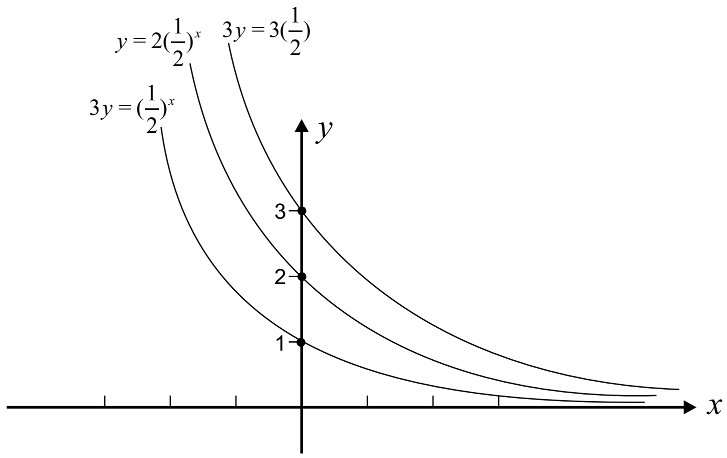
1.



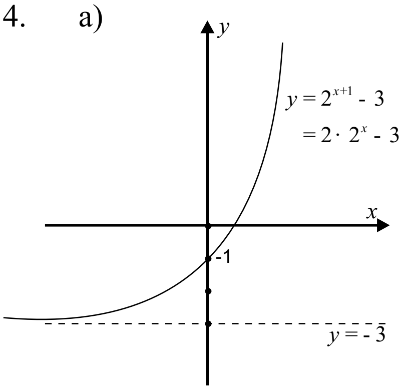
2.



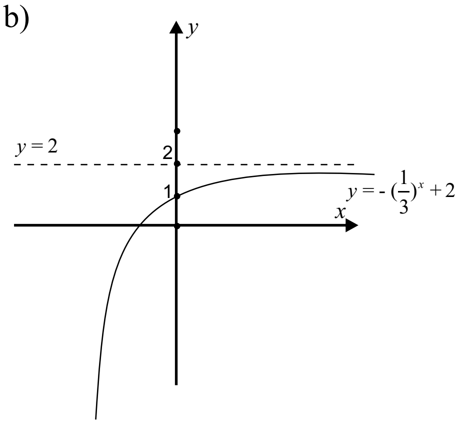
3.



4. a)



b)



Activity 5

1. a) $5^3 = 125$ means $\log_5 125 = 3$

b) $64 = 8^2$ means $\log_8 64 = 2$

c) $3^{-1} = \frac{1}{3}$ means $\log_3 \frac{1}{3} = -1$

d) $4 = 8\frac{2}{3}$ means $\log_8 4 = \frac{2}{3}$

2. a) $\log_{0,1} 100 = -2$ means $0,1^{-2} = 100$

b) $\log 10 = 1$ means $10^1 = 10$
(Remember when the base is not indicated it is always 10)

c) $\log_a b = c$ means $a^c = b$

d) $\log_2 16 = 4$ means $2^4 = 16$

Activity 6

1. a) Let $\log_2 \frac{1}{16} = x$

$2^x = \frac{1}{16} = 2^{-4}$

$x = -4$

$\log_2 \frac{1}{16} = -4$

b) Let $\log 100 = x$ (base 10)

$10^x = 100 = 10^2$

$x = 2$

$\log 100 = 2$

c) Let $\log_7 49 = x$

$7^x = 49 = 7^2$

$x = 2$

$\log_7 49 = 2$

d) Let $\log_a a = x$

$a^x = a = a^1$

$x = 1$

$\log_a a = 1$

e) Let $\log_a 1 = x$

$a^x = 1 = a^0$

$x = 0$

$\log_a 1 = 0$

$\log_a a = 1$
and
$\log_a 1 = 0$

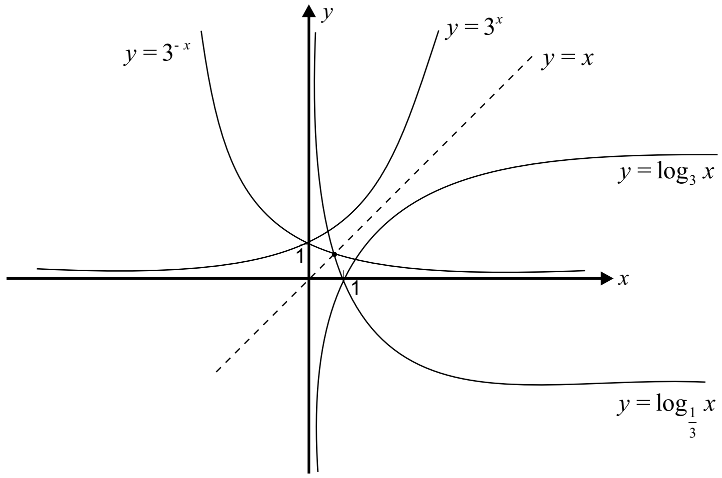
2. a) $\log 2 = 0,3010...$ (base10)

b) $\log 2 = 0,6931...$ (base $e \approx 2,71828...$)

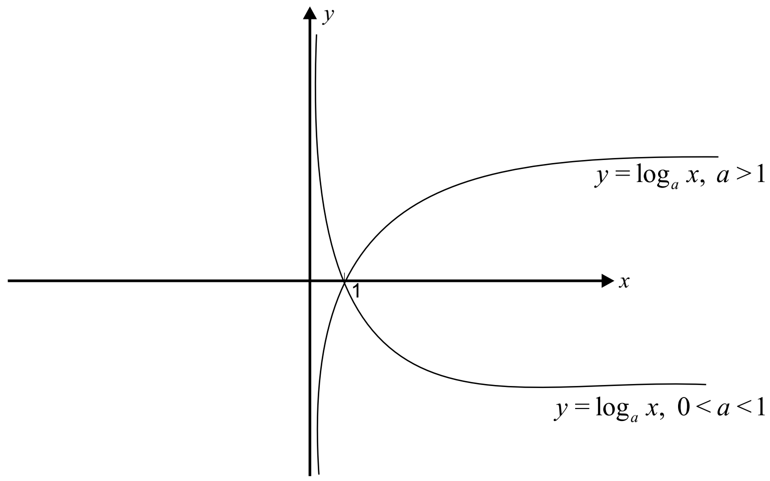
c) $\log_4 2 = 0,5$

d) $\log_{\frac{1}{2}} 2 = -1$

3.



4. The graphs are symmetrical about the x –axis ($y = 0$).



Activity 7

1. a) $2\log 3 + \log 2 - \log 4 = \log 3^2 + \log 2 - \log 4$

$= \log \frac{9 \times 2}{4} = \log \frac{9}{2}$

b) $2\log_a 5 + \log_a 4 + 2\log_a 10$

$= \log_a 5^2 + \log_a 4 - \log_a 10^2$

$= \log_a \frac{25 \times 4}{100} = \log_a \frac{100}{100} = \log_a 1 = 0$

c) $2 - 2\log 5 = 2\log 10 - 2\log 5 = \log 10^2 - \log 5^2$

$= \log \frac{100}{25} = \log 4$

2. a) $\log \frac{abc}{d} = \log a + \log b + \log c - \log d$

b) $\log(6 \times 4) = \log 6 + \log 4 = \log(3 \times 2) + \log 2^2$

$= \log 3 + \log 2 + 2\log 2$

$= \log 3 + 3\log 2$

c) $\log \sqrt{\frac{a}{b}} = \log\left(\frac{a}{b}\right)^{\frac{1}{2}} = \frac{1}{2}\log\left(\frac{a}{b}\right) = \frac{1}{2}\log a - \frac{1}{2}\log b$

3. a) $\log_2 32 - \log_2 8 = \log_2 \frac{32}{8}$

$= \log_2 4 = \log_2 2^2$

$= 2\log_2 2 = 2$

b) $\log 8 + \log 125 - 1 = \log 8 + \log 125 - \log 10$

$= \log \frac{8 \times 25}{10} = \log \frac{1000}{10}$

$= \log 100$

$= \log 10^2$

$= 2\log 10$

$= 2$

c) $\frac{\log 16 - \log 4}{\log 16 + \log 4} = \frac{\log \frac{16}{4}}{\log(16 \times 4)}$

$= \frac{\log 4}{\log 64} = \frac{\log 4}{\log 4^3} = \frac{\log 4}{3\log 4} = \frac{1}{3}$

Activity 8

a) $\log_a x + \log_a 6 = \log_a (x+6)$

$\log_a 6x = \log_a (x+6)$

gives $6x = x + 6$

$5x = 6$

$x = \frac{6}{5}$

b) $3\log_2 x - 3 = 0$

$3\log_2 x = 3$

$\log_2 x = 1$ (or $\log x = \log_2 2$)

$x = 2^1 = 2$ $\therefore x = 2$

c) $2\log x = 4$

$\log x = 2$

$x = 10^2$ or $\log x = 2 = 2\log 10 = \log 10^2$

$x = 100$ $x = 10^2 = 100$

Lesson 5

Activity 1

1. $3^x = 2$

$\log 3^x = \log 2$

$x \log 3 = \log 2$

$x = \frac{\log 2}{\log 3} = 0,63$

2. $5^{-x} = 12$

$\log 5^{-x} = \log 12$

$-x \log 5 = \log 12$

$x = \frac{-\log 12}{\log 5} = -1,54$

3. $(\frac{1}{2})^{x+1} = 5$

$(x+1)\log(0,5) = \log 5$

$x+1 = \frac{\log 5}{\log 0,5} = -2,32$

$x = -3,32$

4. $(1,12)^n = 1,315$

$n\log 1,12 = \log 1,315$

$n = \frac{\log 1,315}{\log 1,12} = 2,42$

5. $\frac{260[1-1,02-n]}{0,02} = 3\ 600$

$1-1,02^{-n} = \frac{3600 \text{ x } 0,02}{260}$

$1,02^{-n} = 1 - \frac{3600 \text{ x } 0,02}{260}$

$-n\log(1,02) = \log[1 - \frac{3600 \text{ x } 0,02}{260}]$

$n = \frac{-\log[1-\frac{3600 \text{ x } 0,02}{260}]}{\log 1,02} = 16,37$

Activity 2

1. $F_v = P_v(1+i)^n$ where $F_v = 1\ 000,\ Pv = 500,\ i = \frac{0,06}{12} = 0,005$

$1\ 000 = 500(1+0,005)^n$

$2 = 1,005n$

$n = \frac{\log 2}{\log 1,005} = 138,98 \text{ months} = 11,58 \text{ years}$

$= 11 \text{ years, } 212 \text{ days } (0,58 \cdot 365 \text{ days})$

2. We must find n given $n = 500,\ i = \frac{0,035}{12},\ F_v = 50\ 000$

$F_v = \frac{x[(1+i)^n - 1]}{i}$

$50\ 000 = \frac{500[(1+\frac{0,035}{12})^n - 1]}{\frac{0,035}{12}}$

$(1+\frac{0,035}{12})^n = \frac{50\ 000 \text{ x } \frac{0,035}{12}}{500} + 1$

$n = \frac{\log[\frac{100 \text{ x } 0,035}{12} + 1]}{\log[1+\frac{0,035}{12}]} = 87,88 \text{ months}$

i.e. 7 years 4 months

3. $F_v = P_v(1-i)^n$

$F_v = 0,15P_v,\ i = 0,20$

$0,15P_v = P_v(1-0,2)^n$

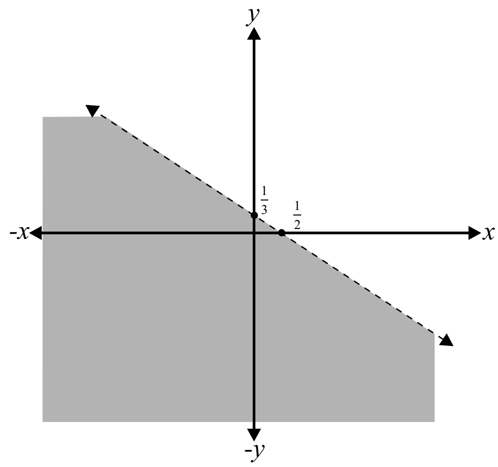
$0,8^n = 0,15$

$n = \frac{\log 0,15}{\log 0,8} = 8,5$ years

Lesson 6

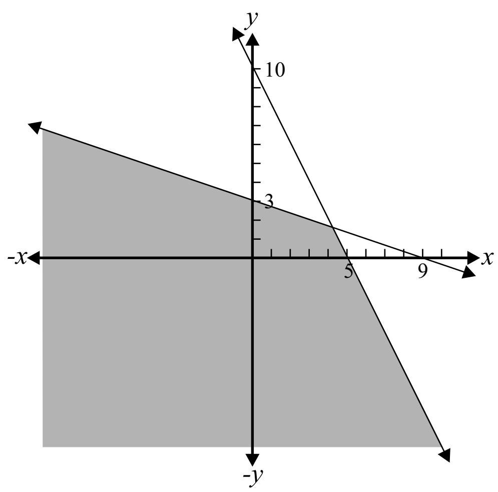
Activity 1

a)



The line is dotted because the inequality is < and not ≤. The points on the line are not included.

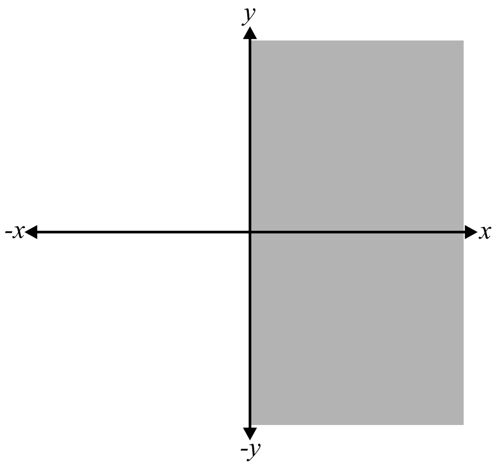
b)



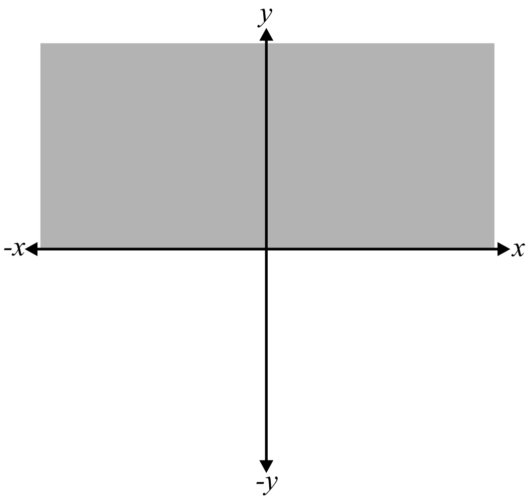
Well done, if you completed this activity successfully. However, if you had problems, then you should go back to revise Lesson 2 of Unit 2.

Activity 2

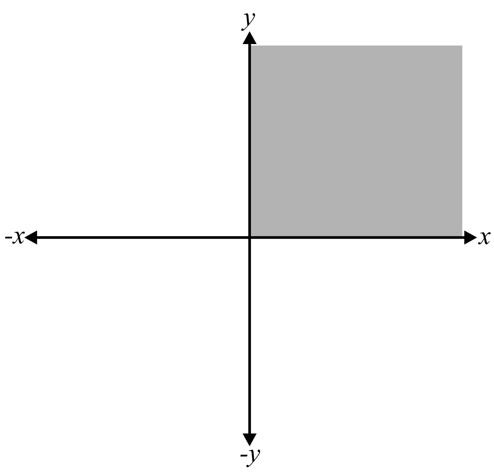
$x \geq 0$ is the region on the right-hand side of the y –axis:



$y \geq 0$ is the region above the x –axis:



Therefore if we combine x ≥ 0 and $y \geq 0$ we get this region:



Only shade the region which satisfies all the inequalities.

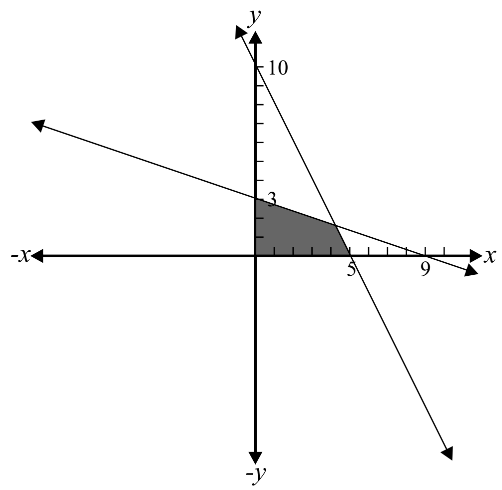
Thus the feasible region for the four inequalities:

$y \leq x + 2$

$y \geq -2x - 1$

$y \geq 0$

$x \geq 0$ is the shaded region shown on the graph alongside.



Activity 3

1. Let x be the number of tablets and y the number of capsules the patient takes a day. Then the patient will take $3x + 4y$ units of vitamin A a day. This must be greater than or equal to 15 units.

   Therefore the inequality is: $3x + 4y \geq 15$, where x and y are integers with $x \geq 0, y \geq 0$.

2. Let x be the number of goats and y the number of sheep the farmer buys. The total cost of the goats and sheep is $200x + 250y$. The maximum amount the farmer can spend on the goats and the sheep is R1 000.

   Therefore $200x + 250y \leq 1\,000$

   $4x + 5y \leq 20$ (divide by 50)

   x, y integers with $x \geq 0, y \geq 0$.

Activity 4

1. Let x and y be the numbers of cheap and expensive seats respectively. Firstly $x \geq 0$ and $y \geq 0$. Then, since the total number of seats must be at most 400, we have: $x + y \leq 400$

   The total space used by the seats must be at most 600m^2.
   Therefore $x + 2y \leq 600$

   The other part of the problem that deals with the allocation of seats to give the most money will be treated in the next section of this lesson.

2. For x kg of food A, there will be $4x$ units of protein and $5x$ units of starch. For y kg of food B, there will be $6y$ units of protein and $3y$ units of starch. The daily intake of protein should be greater than or equal to 16.

Therefore $4x+6y\geq 16$
or $2x+3y\geq 8$

The daily intake of starch should be greater than or equal to 11.
Therefore $5x+3y\geq 11$

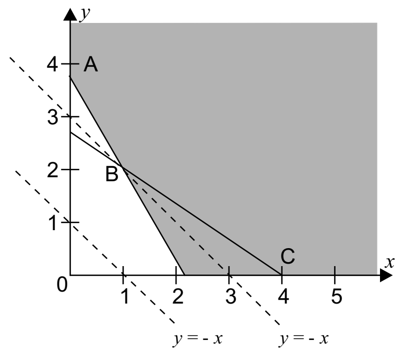
Remember we have to consume food A and food B to get the minimum amount of protein and starch each day so $x \geq 0$ and $y \geq 0$.

Activity 5

1. The first part of the question was dealt with in Activity 4, question 2. We had these inequalities; $2x + 3y \geq 8$ and $5x + 3y \geq 11$. Also x and y cannot be negative. Therefore $x \geq 0$ and $y \geq 0$. This gives us the following sets of inequalities:

$$2x+3y\geq 8$$
$$5x+3y\geq 11$$
$$x\geq 0$$
$$y\geq 0$$

The feasible region or the graph is as shown alongside:



The objective function is the function which involves the weight of food eaten. Let W be the weight of food eaten. Then the objective function is:

$$W=x+y$$

therefore $y = -x + W$ which is the dotted line on the graph. As you move the line across, the vertices of the feasible region are:

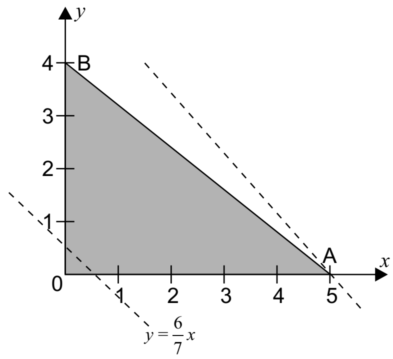
A$(0;\frac{11}{3})$ and B(1;2) and C(4;0).

To get the least weight of food eaten, we have to substitute the values of x and y obtained at the vertices A, B and C into the objective function and choose the ones that give the least value of W.

This gives $3\frac{2}{3}$ kg at A, 3 kg at B and 4 kg at C. The least weight of food eaten is therefore 3 kg. This occurs at B when 1 kg of food A ($x = 1$) and 2 kg of food B ($y = 2$) are eaten.

2. We solved the first part in question 2 of Activity 3. We had the inequality $4x + 5y \leq 20$ where x is the number of goats and y the number of sheep the farmer buys. Now both x and y cannot be negative. We therefore have the additional constraints $x \geq 0$ and $y \geq 0$.

The feasible region is given in the graph below.



The objective function is the function which involves the profit.
Let P be the profit the farmer can make.
Then

$$P = 60x + 70y$$

which can be written as $70y = -60x + P$

$$y = -\frac{60}{70}x + \frac{P}{70}$$

Therefore the gradient of the objective function is $-\frac{6}{7}$.

Let's include this on the graph on the previous page and move it across the feasible region until we reach the maximum point which is A(5;0). When we substitute this into
P = 60x + 70y
P = 6(5) + 70(0)
P = R300

We can check point B(0;4) which is also in the feasible region to make sure that we are correct.
P = 60x + 70y
P = 60(0) + 70(4)
P = R280

It is obvious that the farmer will want the profit of R300 so he will need to buy 5 goats and no sheep to get maximum profit.

Feedback to Self-Check Exercises

Lesson 1

1. To answer this question, we will use our understanding of a general term of an arithmetic sequence to write the terms that are given.

a) $T_5 = a + 4d = -1$ ①

$T_{11} = a + 10d = -19$ ②

This gives us two simultaneous equations with two unknowns. We will use the method of adding or subtracting for solving the simultaneous equations.

② − ①: $6d = -18$

Therefore, $d = -3$

Substitute for d in ①: $a + 4(-3) = -1$

Therefore, $a = 11$

The first three terms are 11; 8; 5.

b) $T_4 = a + 3d = -4$ ①

$T_7 = a + 6d = -6m - 16$ ②

② − ①: $3d = -6m - 12$

Therefore, $d = -2m - 4$

Substitute for d in ①: $a + 3(-2m - 4 = -4$

$a - 6m - 12 = -4$

Therefore, $a = 6m + 8$

The first three terms are $6m + 8$; $4m + 4$; $2m$.

2. In this case we will use our understanding of the general term of a geometric sequence. We will then use the method of dividing for solving the simultaneous equations.

a) $T_3 = ar^2 = 3$ ①

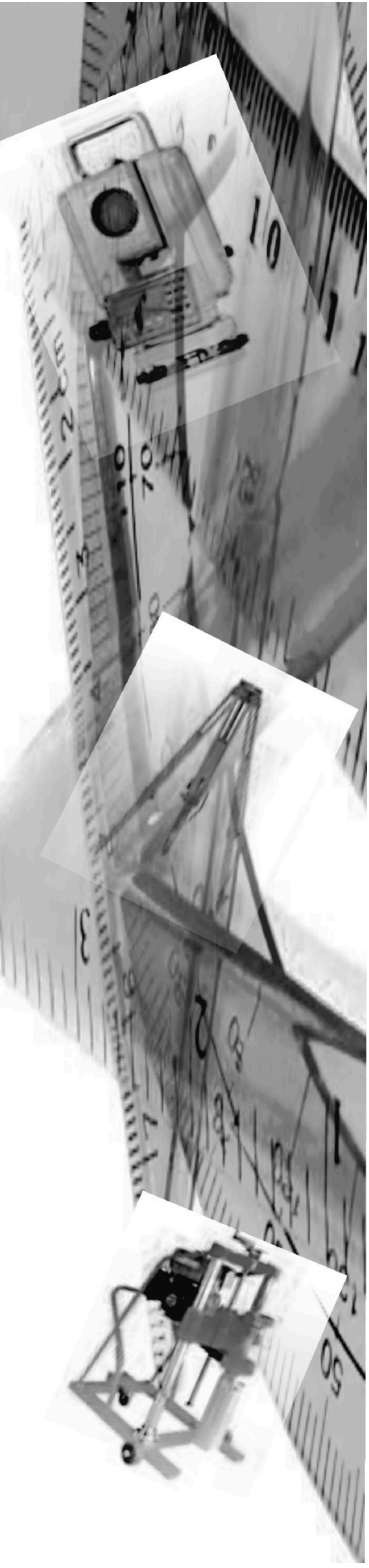
$T_6 = ar^5 = -\frac{1}{9}$ ②

$\frac{ar^5}{ar^2} = -\frac{1}{9} \cdot \frac{1}{3}$ ② ÷ ①

$r^3 = -\frac{1}{27}$

$r = -\frac{1}{3}$

The first three terms are 27; –9; 3.



b) $T_5 = ar^4 = \frac{27}{7}$ ①

$T_8 = ar^7 = \frac{729}{7}$ ②

$\frac{ar^7}{ar^4} = \frac{729}{7} \times \frac{27}{7}$ ②÷①

$r^3 = 27$

$\therefore\ r = 3$

The first three terms are $\frac{1}{27};\ \frac{1}{7};\ \frac{3}{7}.$

c) $T_9 = ar^8 = x^9 + 2x^8b$

$T_{13} = ar^{12} = x^{13} + 2x^{12}b$

When I am given terms of a sequence in this form, I start by factorising the terms, in order to simplify how they look.

Then, $ar^8 = x^9 + 2x^8b = x^8(x+2b)$

$ar^{12} = x^{13} + 2x^{12}b = x^{12}(x+2b)$

$\frac{ar^{12}}{ar^8} = \frac{x^{12}(x+2b)}{x^8(x+2b)}$

$\therefore\ r^4 = x^4$

$r = x$ or $r = -x$

$ax^8 = x^8(x+2b)$ or $a(-x)^8(x+26)$

$\therefore\ a = x+2b$ in both cases

The first three terms are $x+2b;\ x^2+2xb;\ x^3+2x^2b$
or $x+2b;\ -x^2-2xb;\ x^3+2x^2b.$

3. In all the questions we will use the formula for the general term of an arithmetic sequence, $T_k = a + (k-1)d$.

a) $a = 1, d = 4$

$T_k = a + (k-1)d$

$T_5 = a + 4d = 1 + 4 \text{ x } 4 = 17$

$T_{50} = a + 49d = 1 + 49 \text{ x } 4 = 1 + 196 = 197$

b) $a = -22, d = 6$

$T_k = a + (k-1)d$

$T_5 = a + 4d = -22 + 4 \text{ x } 6 = -22 + 24 = 2$

$T_{50} = a + 49d = -22 + 49 \text{ x } 6 = -22 + 294 = 272$

c) $a = 100$

$$T_k = a + (k-1)d$$
$$T_4 = a + 3d$$
$$85 = 100 + 3d$$
$$3d = 85 - 100 = -15$$
$$d = -5$$
$$T_5 = a + 4d = 100 + 4 \times (-5) = 100 - 20 = 80$$
$$T_{50} = a + 49d = 100 + 49 \times (-5) = 100 - 245 = -145$$

d) $a = T_1 = x$

$$T_2 = x - y$$
$$T_2 - T_1 = d = -y$$
$$T_5 = a + 4d = x + (-4y) = x - 4y$$
$$T_{50} = a + 49d = x - 49y$$

4. In all the questions we will use the formula for the general term of a geometric sequence, $T_k = ar^{k-1}$.

a) $a = T_1 = 1$

$T_2 = ar = 3$

Therefore, $r = 3$

$$T_k = ar^{k-1}$$
$$T_5 = ar^4$$
$$= 1 \times 3^4 = 81$$
$$T_{50} = ar^{49}$$
$$= 1 \times 3^{49}$$
$$= 2{,}392\ 993\ 3 \times 10^{23}$$

b) $a = T_1 = x + y$

$T_2 = ar = x^2 - y^2 = (x + y)$

Therefore, $r = x - y$

$$T_k = ar^{k-1}$$
$$T_5 = ar^4$$
$$= (x + y)(x - y)^4$$
$$T_{50} = ar^{49}$$
$$= (x + y)(x - y)^{49}$$

It is not necessary to work out these expressions to the full. A factorised answer is usually tidier than an answer that has been worked out in full.

c) $a = T_1 = 1$

$T_2 = ar = \frac{1}{2}$

$\therefore\ r = \frac{ar}{a} = \frac{\frac{1}{2}}{1} = \frac{1}{2}$

Thus,

$T_5 = ar^4 = 1 \times \left(\frac{1}{2}\right)^4 = \frac{1}{16}$

$T_{50} = ar^{49} = 1 \times \left(\frac{1}{2}\right)^{49} = 2^{-49}$

I am sure you are aware of the form of the last answer. Sometimes it is not compulsory to work out the answer further. You may leave it in exponential form.

d) $T_8 = ar^7 = 3 \times 10^9$

$T_{12} = ar^{11} = 3 \times 10^{13}$

Therefore, $r^4 = 10^4$

$r = 10$

$a(10)^7 = 3 \times 10^9$

$\therefore a = \frac{3 \times 10^9}{10^7}$

$= 3 \times 10^2$

Thus,

$T_5 = ar^4 = 3 \times 10^2 \times 10^4 = 3 \times 10^6$

$T_{50} = ar^{49} = 3 \times 10^2 \times 10^{49} = 3 \times 10^{51}$

5. Since $2; x; y$ is an arithmetic sequence

$x - 2 = y - x$

$\therefore y = 2x - 2$ ①

Since $2; x-1; y$ is a geometric sequence

$\frac{x-1}{2} = \frac{y}{x-1} (x \neq 1)$

$\therefore (x-1)^2 = 2y$ ②

Substituting ① into ②:

$(x-1)^2 = x(2x-2)$

$x^2 - 6x + 5 = 0$

$(x-1)(x-5) = 0$

$x = 1$ or $x = 5$

But $x \neq 1$ from ②

$\therefore\ x = 5$

so $y = 8$ from ①

(Note: $x = 1$ and $y = 0$ do satisfy the statement of the problem if we allow the common ratio of the geometric sequence to be 0, but then the formula $r = \frac{T_{k+1}}{T_k}$ breaks down because we end up with $\frac{0}{0}$, which is not defined.)

6. a) $a = 3, d = 8$

Let 963 be the nth term of the sequence.

Therefore, $963 = a + (n-1)d$

$$963 = 3 + (n-1)8$$
$$= 8n - 5$$
$$968 = 8n$$

Thus, $n = 121$

963 is the 121st term of the sequence.

b) When we defined a sequence, we said that n must be a natural number. This is very important. We will answer this question using that understanding.

$a = 7, r = 2$

Let 546 be the nth term of the sequence, then

$546 = 7.2^{n-1}$ $(T_n = ar^{n-1})$

$\therefore 2^{n-1} = 78$

But 78 is not a power of 2 $(2^6 = 64$ and $2^7 = 128)$ so n is not a natural number ($n \approx 5{,}29$).

$\therefore$ 546 is not a term of the sequence.

c) The odd numbers are an arithmetic sequence with

$a = 1,\ d = 2,\ n = 50$

$$T_{50} = a + (n-1)d = 1 + (50-1)2$$
$$= 1 + 98$$
$$= 99$$

The 50th odd number is 99.

Lesson 2

1. The denominator in the formula is $r - 1$. We cannot divide by zero, which means that we cannot allow a situation where the denominator in any fraction is zero. That is why r cannot be equal to 1.

2. a) This is an arithmetic series with $a = 3, d = 5$

$$63 = a + (n-1)d$$
$$= 3 + (n-1)5$$
$$= 5n - 2$$
$$65 = 5n$$

Therefore, n = 13

The general term, $T_n = 5n - 2$

$$\sum_{n=1}^{13}(5n-2) = \frac{n}{2}[2a + (n-1)d]$$
$$= \frac{13}{2}[2(3) + (13-1)5]$$
$$= \frac{13}{2}[6 + 60]$$
$$= 13 \text{ x } 33$$
$$= 429$$

b) This is a geometric series with $a = 3, r = -2, n = 10$

The general term, $T_n = ar^{n-1} = 3(-2)^{n-1}$

$$\sum_{n=1}^{10} 3(-2)^{n-1} = \frac{a(r^n - 1)}{r - 1}$$
$$= \frac{3[(-2)^{10} - 1]}{-2-1}$$
$$= \frac{3(1024 - 1)}{-3}$$
$$= -1\ 023$$

c) This is an infinite geometric series with $a = 9, r = \frac{2}{3}$

$$T_n = ar^{n-1} = 9\ \frac{2}{3}\ ^{n-1}$$
$$\sum_{n=1}^{00} 9\left(\frac{2}{3}\right)^{n-1} = \frac{a}{1-r}$$
$$= \frac{9}{1 - \frac{2}{3}}$$
$$= 27$$

3. a) $\sum_{i=1}^{n} 1$

$= 1 + 1 + 1 + ... + 1$ (n terms)

$= n$

b) $\sum_{i=1}^{n} i$

$= 1 + 2 + 3 + ... + n$

$= \frac{n}{2}\{1 + n\}$

$= \frac{n(n+1)}{2}$

c) $\sum_{i=1}^{n}(2i-1)$

$=1+3+...+(2n-1)$

$=\frac{n}{2}\{1+(2n-1)\}$

$=n^2$

4. 0,343 434...

$=\frac{34}{100}+\frac{34}{10\ 000}+\frac{34}{1\ 000\ 000}+...$

This is an infinite geometric series with a $a=\frac{34}{100}$ and $r=\frac{1}{100}$

$\therefore\ 0,343\ 434...=\frac{a}{1-r}$

$=\frac{\frac{34}{100}}{1-\frac{1}{100}}$

$=\frac{34}{99}$

Lesson 3

1. a) $F_v = P_v(1+in) = 1\ 200(1+0,045\text{ x }5) = \text{R}1\ 470,00$

b) $F_v = P_v(1+i)^n = 1\ 200(1+0,045)^5 = \text{R}1\ 495,42$

c) $F_v = P_v(1+\frac{i}{4})^{4n} = 1\ 200(1+0,01125)^{20} = \text{R}1\ 500,90$

d) $F_v = P_v(1+\frac{i}{365})^{365n} = 1\ 200(1+\frac{0,045}{365})^{365\text{ x }5} = \text{R}1\ 502,77$

2. a) $j = i = 4,5\%$ p.a.

b) $j = i = 4,5\%$ p.a.

c) $1+j=(1+0,011\ 25)^4$

$j = 0,045\ 77 = 4,58\%$ p.a.

d) $1+j=(1+\frac{0,045}{365})^{365}$

$j = 0,046\ 0 = 4,6\%$ p.a.

3. The formula for F_v gives us the amount immediately after the last payment (1st December 2016). The money then earns interest for an extra month.

$\therefore$ Final amount at the end of December 2016 is:

$A = F_v(1+\frac{0,04}{12)})^{12} \quad (x=100,\ i=\frac{0,04}{12},\ n=5\text{ x }12=60)$

$=\frac{100[(1+\frac{0,04}{12})^{60}-1]}{\frac{0,04}{12}}\text{ x }(1+\frac{0,04}{12})^{12}$

$=\text{R}6\ 900,01$

4. a) We must find P_v given $x = 3\ 500,\ i = \frac{0,105}{12},\ n = 20 \times 12 = 240$

$$P_v = \frac{3\ 500[1-(1+\frac{0,105}{12})^{-240}]}{\frac{0,105}{12}} = R350\ 567,96$$

b) Outstanding balance = P_v of remaining payments

$$P_v = \frac{3\ 500[1-(1+\frac{0,105}{12})^{-120}]}{\frac{0,105}{12}} = R259\ 384,15$$

c) $\frac{350\ 567,96 - 259\ 384,15}{350\ 567,96} = 0,26 = 26\%$ has been paid off after half the period! This is because the initial payments are paying off mostly interest and no capital.

5. a) The bank is **buying** the currency

so £270 = 270 x 11,167 3 = R3 015,17

and ¥1 200 = $\frac{1\ 200}{11,113\ 2}$ = R107,98

The bank gives you R3 123,15

b) The bank is **selling** the currency

The new exchange rates are:

\$1 = R7,43 (7,31 + 0,12; a dollar costs 12c **more**)

R1 = 0,9090 Pula (previously R1,00 = 0,8695 Pula;

i.e. 1 Pula = $\frac{1}{0,8695}$ = R1,15, so now 1 Pula = R1,10

$\therefore$ R1 = $\frac{1}{1,10}$ = 0,9090 Pula)

Thus R1 000 = $\frac{1\ 000}{7,43}$ = \$134,59

and R1 000 = 1 000 x 0,9090 = 909 Pula

c) 1st Sept: R10 000 = 10 000 · 6,2484 = 62484 Rupees

Remaining Rupees = 62 484 – 50 000 = 12 484 Rupees

The buying price was R1,00 = 6,6780 Rupees

i.e. 1 Rupee = $\frac{1}{6,6780}$ = R0,1497

so new buying price is R0,1497 + R0,05 = R0,1997

$\therefore$ for 12 484 Rupees the bank gives you
12 484 × 0,1997 = R2 493,05

Remember:
When the Rand weakens, you get **more** Rand.
When the Rand strengthens, you get **fewer** Rand.

Lesson 4

1. a)
$$2x^{\frac{1}{2}} = 4$$
$$x^{\frac{1}{2}} = 2$$
$$x = 4$$

b)
$$27^x \text{ x } 9^{x-2} = 1$$
$$3^{3x} \text{ x } 3^{2(x-2)} = 3^0$$
$$3^{3x+2x-4} = 3^0$$
$$5x - 4 = 0$$
$$x = \frac{4}{5}$$

c)
$$2^{2x-1} = 8$$
$$2^{2x-1} = 2^3$$
$$2x - 1 = 3$$
$$2x = 4$$
$$x = 2$$

d)
$$4^x = 64$$
$$4^x = 4^3$$
$$x = 3$$

2. a) $x = 4^3$ b) $3^5 = 243$

c) $10^y = 0,000\,1$ d) $8^{\frac{2}{3}} = 4$

3. a) $\log 12$ b) $\log 3$

c) $\log_a 3$ d) $\log xy$

4. a) 6 b) 3

c)
$$\log(3x + 21) - \log(2x + 1) = \log 8$$
$$\log \frac{3x + 21}{2x + 1} = \log 8$$
$$\text{or } \frac{3x + 21}{2x + 1} = 8$$
$$3x + 21 = 8(2x + 1)$$
$$3x + 21 = 16x + 8$$
$$13x = 13$$
$$x = 1$$

d) $\log_3(x-4)+\log_3(x-2)=1$

$$\log_3(x-4)(x-2)=1$$

$$(s-4)(x-2)=3^1$$

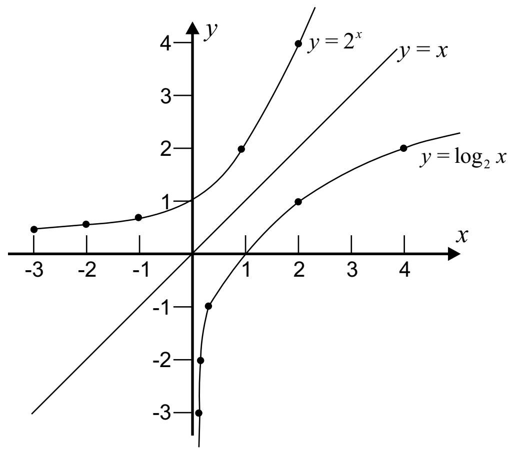
$$x^2-6x+8=3$$

$$x^2-6x+5=0$$

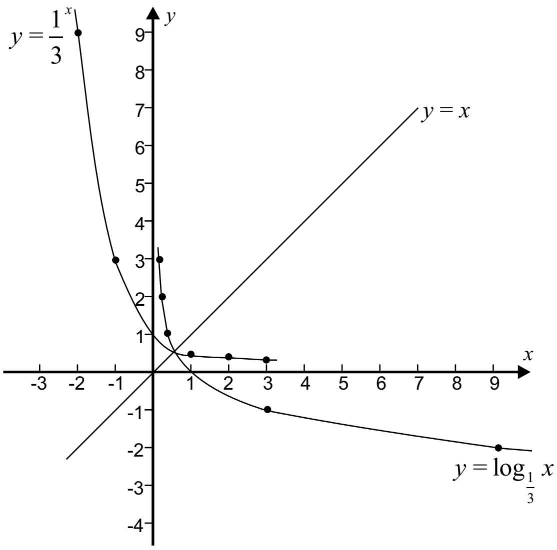
$$(x-1)(x-5)=0$$

$$x=1 \text{ or } 5$$

5. a)

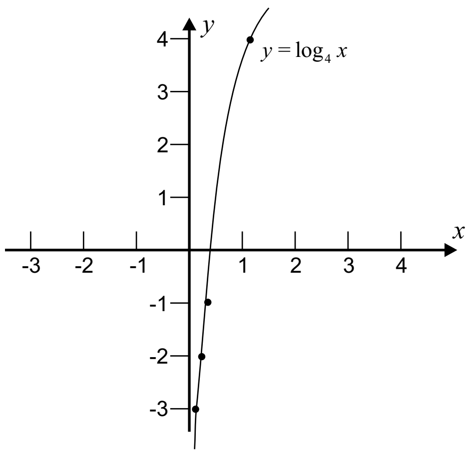


b)



6.

x	$\frac{1}{16}$	$\frac{1}{4}$	1	4	16
y	–2	–1	0	1	2



Lesson 5

1. a)

$$2x = 12$$
$$x \log 2 = \log 12$$
$$x = \frac{\log 12}{\log 2}$$
$$= 3,58$$

b)

$$3 \cdot 5^{x+1} = 125$$
$$5x + 1 = \frac{125}{3}$$
$$(x+1)\log 5 = \log(\frac{125}{3})$$
$$x + 1 = \frac{\log(\frac{125}{3})}{\log 5}$$
$$= 2,32$$
$$x = 1,32$$

c)

$$3^{-x+2} = 18$$
$$(-x+2)\log 3 = \log 18$$
$$-x + 2 = \frac{\log 18}{\log 3}$$
$$= 2,63$$
$$x = -0,63$$

2. a) $1 + j = (1 + \frac{0,065}{12})^{12} \therefore j = 0,066\ 97 \approx 6,7\%$

b) Method 1: $2P_v = P_v(1 + \frac{0,065}{12})^n$ (n in months)

$$(1 + \frac{0,065}{12})n = 2$$
$$n = \frac{\log 2}{\log(1 + \frac{0,065}{12})} = 128,31 \text{ months}$$
$$= 10,69 \text{ years}$$

Method 2: Using the effective interest rate from a)

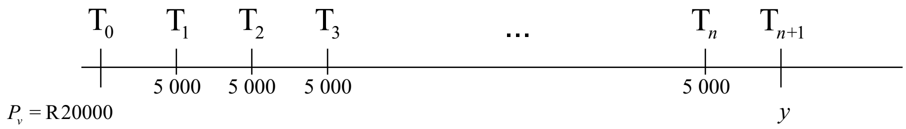
$$2P_v = P_v(1+0,066\ 97)^n \quad (n \text{ in years})$$
$$(1,066\ 97)^n = 2$$
$$n = \frac{\log 2}{\log 1,066\ 97} = 10,69 \text{ years}$$

3. Outstanding balance = P_v of remaining payments

$$P_v = 10\ 907,51,\ x = 1000,\ i = \frac{0,18}{12} = 0,015,\ n = \text{no. of payments}$$

$$10\ 907,51 = \frac{1\ 000[1-(1+0,015)^{-n}]}{0,015}$$
$$(1,015)^{-n} = 1 - \frac{10\ 907,51 \text{x}\ 0,015}{1\ 000}$$
$$n = \frac{-\log[1-\frac{10907,51\text{x}\ 0,015}{1000}]}{\log 1,015} = 12 \text{ more payments}$$

4.



a) Suppose n payments of R5 000 p.m.

$$P_v = 30\ 000,\ n = 5\ 000,\ i = \frac{0,14}{12}$$

$$30\ 000 = \frac{5\ 000[1-(1+\frac{0,14}{12})^{-n}]}{\frac{0,14}{12}}$$

$$(1+\frac{0,14}{12})^{-n} = 1 - \frac{30\ 000 \text{x} \frac{0,14}{12}}{5\ 000} = 1 - 0,07$$

$$n = \frac{-\log(1-0,07)}{\log(1+\frac{0,14}{12})} = 6,256...$$

Thus 6 payments of R5 000 are required.

b) Suppose the payment is y, then

$$30\ 000 = \frac{5\ 000[1-(1+\frac{0,14}{12})^{-6}]}{\frac{0,14}{12}} + y(1+\frac{0,14}{12})^{-7}$$

$$\therefore y = [30\ 000 - \frac{5\ 000[1-(1+\frac{0,14}{12})^{-6}]}{0,14}](1+\frac{0,14}{12})^{7}$$

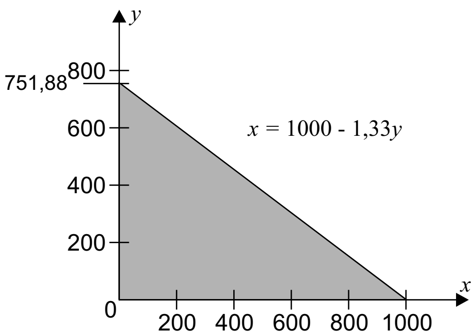
$$= \text{R}1\ 288,34$$

Lesson 6

1. a) $1,5x + 2y \leq \text{R}1500$

b) $x \geq 0$
$y \geq 0$

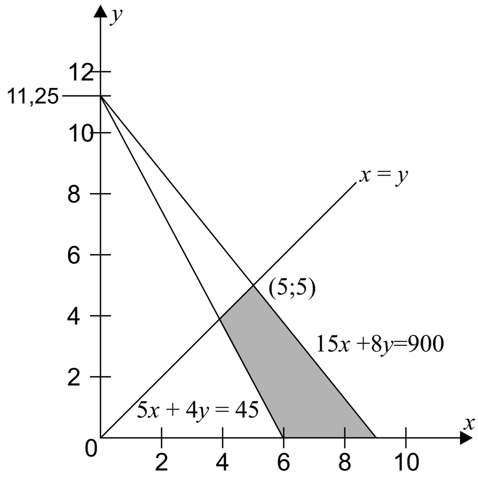
c) $x \leq \frac{\text{R}1500 - 2y}{1,5} = 1000 - 1,33y$



d) (i) $\text{P} = \text{R}0,4x + \text{R}0,5y$
(ii) Maximum profit $\text{P} = \text{R}0,4\,(1\ 000) = \text{R}400$

2. a) $150\text{x} + 80\text{y} \geq 900$ parcels
$50\text{x} + 40\text{y} \leq \text{R}450$
$x \leq y \qquad x \geq 0 \qquad y \geq 0$

b)



c) The objective function is: $n = x + y$

Thus the least number of journeys is $x = 6$ and $y = 0$.